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Diversification and Financial Stability

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Abstract

The recent credit crisis of 2007/08 has raised a debate about the so-called *knife-edge* properties of financial markets. The paper contributes to the debate shedding light on the controversial relation between risk-diversification and financial stability. We model a financial network where assets held by borrowers to meet their obligations, include claims against other borrowers and securities exogenous to the network. The balance-sheet approach is conjugated with a stochastic setting and by a mean-field approximation the law of motion of the system's fragility is derived. We show that diversification has an ambiguous effect and beyond a certain levels elicits financial instability. Moreover, we find that risk-sharing restrictions create a socially preferable outcome. Our findings have significant implications for future policy recommendation.

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1 Introduction

The recent financial crisis is attracting increasing attention on the *knife-edge* properties of the financial system and its optimal architecture (see e.g., Gai and Kapadia, 2007; Haldane and May, 2011; Nier et al., 2007). In a nutshell, this means that within a certain range, financial interconnections serve as shock-absorber (i.e., connectivity engenders robustness and risk-sharing prevails). But beyond the tipping point, interconnections serve as shock-amplifier (i.e., connectivity engenders fragility and risk-spreading prevails). It has been argued that these *knife-edge* dynamics, can essentially be explained by two structural features over of the financial network. They are complexity on the one hand, and homogeneity on the other (Haldane, 2009). The paper contributes to this discussion by shedding light on the relation between diversification and financial stability. It investigates how diversification affects the probability of failure, both from an individual and a systemic perspective, when financial institutions are (1) interlinked at

the balance-sheet level in a network of obligations and (2) invest in activities external to the financial network. Until recently, many theoretical models of finance pointed towards the stabilizing effects of a diversified (i.e., dense) financial network, (Allen and Gale, 2001). Some recent work has started to challenge this view, investigating conditions under which diversification may have ambiguous effects; e.g., Battiston et al. (2009); Brock et al. (2009); Ibragimov et al. (2011); Ibragimov and Walden (2007); Stiglitz (2010); Wagner (2009).

In a similar spirit as Shin (2008, 2009), we model a financial network composed of leveraged risk-averse financial institutions (hereafter "banks"), which invest in two classes of assets. The first class consists of obligations issued by other banks in the network. [1] The second class represents assets external to the financial network, which may include securities on, e.g., real-estate and other real-economy related activities, or loans to firms and households (hereafter "external assets"). We assume that external assets may be in a downtrend (hereafter, *downturn*) with probability p, or in an uptrend (hereafter, *upturn*), with probability $1 - p$. In this uncertain world, it is assumed that banks have an incentive to diversify across external assets.[1] Precisely, the strategy consists of choosing an optimal portfolio that maximizes their expected Mean-Variance (MV) utility function. For the sake of simplicity it is assumed that banks adopt a "naive" strategy which involves holding an equally weighted portfolio of external assets. Two important assumptions of the model are as follows: first, the up(down)turn is *persistent* i.e., approximately constant during a given period Δt; second, due to information-gathering and transaction costs, the diversification strategy is assumed to be *static* for a given period Δt.[2] These two assumptions imply that, based on the expectation about the future trend of external assets, each bank sets up a diversified portfolio which will be held for a finite arbitrary interval Δt.

A first contribution of the paper is to show that banks' default probability increases with diversification in case of a downturn and decreases in case of an upturn. There is a simple intuition for this effect in terms of the Sharpe ratio which allows to compare different investment returns per unit of risk. When assets have positive expected cash-flows the Sharpe ratio is positive as well. Then it is desirable to combine them in a well diversified portfolio since diversification lowers portfolio's volatility and in doing so, increases its Sharpe ratio. In contrast, when assets have negative expected cash-flows, the Sharpe ratio is negative. Then, a well diversified portfolio is no longer desirable. From the first finding, it follows that the gap between expected MV utility in upturn, w.r.t. expected MV utility in downturn, increases

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[1] Although in our simplified economy, the securities in this class are priced depending on the financial health of the obligors, in reality, the debt-claim structure of the financial system is much more intricate. To better understand this view, we need to achieve a clear understanding of the building blocks of financial system, mostly in the presence of amplification mechanism stemming from leverage and other factors, (see Trichet, 2009). The banks' asset values depend on the payoffs received from the claims on other banks in which they have invested. But the value of these issuing banks depends, in turn, on the financial health of yet other banks in the system. Moreover, linkages between banks can be cyclical. See Eisenberg and Noe (2001).

[1] Generally, this practice is desirable since it allows banks to promise investors a lower repayment or increase the amount of debts. See Allen et al. (2010).

[2] In the real-world, portfolio adjustments are costly. In particular, the revision costs sometimes could be greater than the increase of the expected rate of return of the revised portfolio. So, portfolio revision is not always convenient.

with diversification. Thus, diversification generates and amplifies the knife-edge property of the financial network. Moreover, for a given level of diversification, the gap in utility increases with the magnitude of the trend. In other words, we show that diversification is a *double-edged* strategy, in the words of Haldane (2009).

In the hypotheses that (1) banks' portfolios of external assets do not overlap (i.e., they are not positively correlated) and (2) banks are not interconnected (i.e., they do not invest in other banks' obligations), each bank's default is an independent event. The model relaxes both those hypotheses such that the default of a single bank is very likely to have a systemic impact. Similarly to Acharya (2009), the relaxation of the first hypothesis implies that the systemic failure is the joint failure arising from the correlation of returns on asset-side of banks' balance-sheet. Whereas, resembling Ibragimov et al. (2011), by relaxing the second hypothesis, the systemic risk is introduced when banks with limited liability become interdependent by taking position in other banks' obligations. To summarize, in this setting, we find that diversification, even though desirable in upturn periods, tends to induce a systemic failure in downturn periods. This effect is in line with the *robust-yet-fragile* property of "connected" networks (see e.g., May et al., 2008).

On the basis of the first contribution, the paper shows that by assigning a probability p to the downturn event and $1 - p$ to the upturn event, diversification displays a tradeoff w.r.t. the bank's utility. This means that, the expected MV utility exhibits a maximum which corresponds to an intermediate level of diversification. Such a level depends on the probability p of downturn, on the magnitude of the expected profit (loss) and on the values of the other parameters of the model.

In the presence of limited liability, the expected loss of an individual bank is bounded. In contrast, at the system level, either due to direct or indirect exposures, the externalities associated with the failure of interconnected institutions, amplify the expected losses. These externalities reflect the "incremental costs" due to a system collapse which are ultimately borne by taxpayers. In other words, the cost of recovery from a systemic crisis, grows with the number of defaults. Under this condition, we find that the optimal level of diversification for the regulator is always smaller than the one desirable at individual level. This leads us to another finding. Individual banks' incentives favor a financial network that is over-diversified in external assets w.r.t. to the level of diversification that is socially desirable. Therefore, our analysis has some implications for risk management and policies to mitigate systemic risk.

The paper is organized as follows. In Section 2 we introduce the model. Section 3 derives the financial fragility under the mean-field approach. Section 4 moves from the financial fragility to the default probability and the optimal diversification policy. Section 5 investigates the dichotomy between private incentives of diversification and its social effects. Section 6 concludes and considers some extensions.

2 Model

In a similar spirit as Shin H.S.Shin (2008, 2009), we model a system of n financial risk-averse leveraged institutions ("banks"). The balance-sheet identity conceives the equilibrium between asset an liability side, which for the representative bank-$i \in \{1, ..., n\}$, is

$$a_i = \bar{p}_i + e_i , \tag{1}$$

where $\mathbf{a} := (a_1, ..., a_n)^T$ is the column vector of market values of the banks' assets; $\bar{\mathbf{p}} := (\bar{p}_1, ..., \bar{p}_n)^T$ is the column vector of the book-face values of the banks' obligations (e.g., zero-coupon bonds); [3] $\mathbf{e} := (e_1, ..., e_n)^T$ is the column vector of equity values. Notably, each bank asset-side is composed by two investment classes: (*i*) $n - 1$ obligations, each issued by one of the other banks in the system, and (*ii*) m exogenous investment opportunities non related to the financial sector (external assets). Then, the asset-side in (1) becomes

$$a_i := \sum_j^m \mathbf{Z}_{ij} y_j + \sum_j^n \mathbf{W}_{ij} p_j , \tag{2}$$

where

$$p_j := \bar{p}_j / (1 + r_j)^{-t} \tag{3}$$

is the market value of debt issued by bank-j, defined as the discounted value of future payoffs; $\mathbf{y} := (y_1, ..., y_m)^T$ is the column vector of external investments; $\mathbf{Z} := [z_{ij}]_{n \times m}$ is the $n \times m$ weighting matrix of external investments where each entry $z_{i,j}$ is the proportion of activity-j held by bank-i; $\mathbf{W} := [w_{ij}]_{n \times n}$ is the $n \times n$ adjacency matrix where each entry w_{ij} is the proportion of bank-j debt held by bank-i; r_j : is the rate of return used to discount the debt-face value of the obligor-j. In brief, the bank-i manages the following balance-sheet

Assets	Liabilities
$\sum_j^m \mathbf{Z}_{ij} y_j$	$\bar{p}_i$
$\sum_j^n \mathbf{W}_{ij} p_j$	e_i

Kept the adjacency matrix $\mathbf{W} := [w_{ij}]_{n \times n}$ of claims and obligation among banks as given, the objective function of risk-averse banks, is to increase their asset value per unit of risk. This goal can be achieved through the following *management strategy*.

Definition 2.1. *Management Strategy In a one period capital-allocation problem, banks strategy consists in wealth diversification among the investment opportunities offered by the non-financial sector represented by the adjacency matrix* $\mathbf{Z} := [Z_{ij}]_{n \times m}$.

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[3]In particular, $\bar{p}_i$ is the value of the payment at maturity that each obligor-i promised to the lenders. For all the banks in the system, the equation indirectly assumes only one issue of bonds with the same maturity date t.

External assets include e.g., activities related to the real-economy, loans to firms and households, etc., and each of them it is assumed to generates a log-normal cash flows process

$$d\log(Y_j)(t) := dy_j(t) = \mu_j dt + \sigma_j dB_j(t) \quad \forall j = 1, ..., m \,, \tag{4}$$

where $B_j(t)$ is a standard Brownian motion defined on a complete filtered probability space $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}; \mathbb{P})$, with $\mathcal{F}_t = \sigma_y\{B(s) : s \leq t\}$, μ_j is the instantaneous risk-adjusted expected growth rate, $\sigma_j > 0$ is the volatility of the growth rate. We assume correlation across processes $d(B_i, B_j) = \rho_{ij}$ and i.i.d. returns for each of them. The processes in (4) have the relevant property that they preclude instantaneous external assets' distributions from becoming negative at any time $t \geq 0$ and are therefore an economically reasonable assumption.

The results in the next Sections hold under the following main assumption of the model.

Assumption 2.1. (i) *There exists a frictionless market where all debt-securities have the same maturity date and seniority*; (ii) *Divisibility and Not-short selling constraint. All the securities are marketable, perfectly divisible and can be held only in positive quantities*; (iii) *Price Taking Assumption. Securities prices are beyond the influence of investors*; (iv) *Only expected returns and variances matter. Quadratic utility or normally distributed returns*; (v) *Not-paying dividend-coupon assumption. The return on any security is simply the change in its market price during the holding period*; (vi) *Homogeneous expectations*

2.1 Leverage and Balance-Sheet Management

Given the vector $\boldsymbol{\phi} \in \mathbb{R}^{n \times 1}$ of debt-asset ratios, a standard accounting approach is to consider $\phi_i \in [0, 1]$ as the bank's *i* leverage, such that

$$\phi_i = \bar{p}_i / a_i$$

$$= \bar{p}_i / \left(\sum_j^m \mathbf{Z}_{ij} y_j + \sum_j^n \mathbf{W}_{ij} p_j \right) . \tag{5}$$

The balance-sheet equilibrium between sources of finance and investments cash-flows is a precondition[4] to solvency. In fact, the solvency of a bank is indicated by the positive net worth as measured by difference between assets and liabilities (excluding capital and reserves). Then the problem of a failing bank normally emerges through illiquidity. In this perspective, the debt-asset ratio is a leverage indicator resuming the potential insolvency of a bank due to a fragile capital structure.

Definition 2.2. *Bank Fragility*. *The debt-asset ratio ϕ is a leverage index that maps the bank's fragility between zero and one.*

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[4]Necessary but not sufficient.

A leverage ϕ which tends to one is symptomatic of a very fragile bank that is prone to be highly sensitive to external shocks which may force the bank towards a bankruptcy regime. On the other hand, a leverage ϕ close to zero describes a bank with a solid balance-sheet equilibrium between internal and external sources of finance.

In line with a large body of theoretical and empirical literature, the rate of return on future payments depends upon the payor's credit-worthiness (e.g. measured by its leverage), the contractual features (e.g., time to maturity) and the market factors. In our simplified economy, the only two factors explaining the internal rate of return, are the risk-free rate r_f and the indicator of fragility ϕ such that, for each obligor-j the following equation holds true

$$r_j := r_f + \beta\phi_j . \tag{6}$$

The risk-free rate[5] is fixed over time and is in the common knowledge of every institution: $\mathbb{E}(r_f) = r_f$ and $\mathrm{Var}(r_f) = 0$. The parameter β in $\mathbb{R} \in [0, 1]$ can be interpreted as the responsiveness of the rate of return to a firm's financial condition. Finally, ϕ_j is defined as in (5).[6]

An extensive form of the balance-sheet management of the bank can be now derived substituting (6) into (3)[7] and then (3)[8] into (5)

$$\phi_i = \bar{p}_i / \left(\sum_j^m \mathbf{Z}_{ij} y_j + \sum_j^n \mathbf{W}_{ij} \bar{p}_j / \left(1 + r_f + (\beta\phi_j) \right) \right) . \tag{7}$$

Eq. (7) shows a non-linear dependence of the fragility ϕ_i related to the representative bank-i, with the fragilities $\phi_{j=1,...,n}$ of the j-banks, obligors of i. The following feature of our model is then immediate.

Lemma 2.3. *At a macro level, it can be shown that the fragility of the banking system is the positive root of a system of parabolic expressions in $\boldsymbol{\phi}$.*

Proof. See Appendix A. □

Since (7) is too complex to analyze directly under arbitrary network structures, we follow the usual technique of analyzing a mean-field approximation to the given expression (see e.g., Vega-Redondo, 2007).

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[5]The risk-free rate is paid by an asset without any default, timing or exchange rate risk; as such, it is a non-observable theoretical construct. It is usually measured by the rates of return on government securities, which have the lowest risk, in any particular currency.

[6]With the opportune swap between sub-index j and i.

[7]This substitution makes evident that p_i is an inverse function of ϕ_i. Higher leverage implies a lower market price of the claims and a greater internal rate of return.

[8]Since, p_i in its interval of existence, is a monotonic decreasing function in t, without loss of generality, we consider the time interval in (3) to be $t = 1$.

3 Mean Field Analysis

An approximate description of the equilibrium properties of the system is obtained by using a mean-field approximation of (7) as follows. [9] In the model, bank-i has claims against other banks, each of them with fragility ϕ_j. Their individual fragility is replaced by ϕ (i.e., the average fragility over all banks, $\phi_j \approx \frac{1}{n}\sum_i^n \phi_i := \phi$). Let the proportion of claims against the other banks be equal to $1/n$. Namely, $w_{1j} \approx w_{2j}... \approx w_{nj} \approx 1/n$, $\forall i = 1,...,n$. Then, the aggregate value of the claims is replaced by the average market value $\sum_j^n \mathbf{W}_{ij}\mathbf{p}_j \approx \frac{1}{n}\sum_j^n p_j := p$. Assuming $z_{1j} \approx z_{2j}... \approx z_{nj} \approx 1/m$, $\forall j = 1,...,m$, the value of investment in external assets is substituted by the average value of external assets held in portfolio $\sum_j^m \mathbf{Z}_{ij}\mathbf{y}_j \approx \frac{1}{m}\sum_j^m y_j := y$. The book value of promised payments at maturity is assumed to be equal for every bank. Namely, $\bar{p}_i = \bar{p}$ for all $i = 1,...,n$. Then, (7) becomes $\phi_i = \bar{p}/(y + \frac{\bar{p}}{1+r_f+\beta\phi})$. Under the further condition that, the individual bank-i' fragility resembles the average one, we obtain the mean-field approximation $\phi = \bar{p}/(y + \frac{\bar{p}}{1+r_f+\beta\phi})$. Solving for ϕ, we get

$$\phi\left(y + r_f y + \beta\phi y + \bar{p}\right) = \bar{p}(1 + r_f + \beta\phi)$$

$$...$$

$$\phi^2\beta y + \phi\left(yR + \bar{p}(1-\beta)\right) - \bar{p}R = 0 \, .$$

Then, the fragility is a solution of a quadratic expression in ϕ

$$\phi = \frac{1}{2\beta y}\left[\bar{p}(\beta-1) - Ry + \left(4\beta\bar{p}Ry + (\bar{p}(1-\beta) + Ry)^2\right)^{1/2}\right] \, , \tag{8}$$

where $R = 1 + r_f$. Eq. (8), remarks a dependence of ϕ from the aggregated value of external assets/projects which follow a dynamics derived from (4). More precisely, the following proposition holds.

Proposition 1. *Given a system of SDEs in (4), under the mean-field approximation, the bundle of external assets follows the dynamics $dy = \mu_y dt + \sigma_y dB$ with $dB \sim N(0, dt)$.*

Proof. See Appendix A. □

It can be shown (through Itos Lemma), that the process $\{\phi\}_{t\geq 0}$ driving the dynamics of (8) is fully characterized by the underlying process $\{y\}_{t\geq 0}$ in (16). Since y appear in the denominator of (5), a downturn of $\{y\}_{t\geq 0}$ implies an increase of ϕ. While, an upturn of $\{y\}_{t\geq 0}$ causes ϕ to decrease. Thus, the soundness of one bank and the benefit of its diversification in external assets, is conditioned to the up and downturns of

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[9] The mean-field approach is a useful simplification that allows to characterize analytically the behavior of the system under certain assumptions. The main idea is to replace the state variable of each individual with the average state variable across the system. In this approximation, the individual interacts with the "average individual". Clearly, under this condition, some network aspects become unobservable. On the other hand, it has been argued that the financial sector exhibits increasing levels of homogeneity. Many risk management models (e.g., VaR) and investment strategies tend to look and act alike with the consequence that the balance sheets' structure of the banks are homogenized. "[...]*The level playing field resulted in everyone playing the same game at the same time, often with the same ball*" (Haldane (2009).

the non-financial activities into which the bank invests.[10] In the next Sections we will refer to the upturn, when those assets generate a negative cash-flow (on average). Namely, when $\mu_y < 0$. While, we will refer the upturn when they engender a positive cash-flow (on average). Namely, when $\mu_y > 0$. We summarizes this feature of our model as follow.

Assumption 3.1 *The aggregate value of external assets can be in upturn or in downturn such to engenders positive or negative cash-inflows for the holder. Both trends are assumed to be persistent i.e., approximately constant during a given period Δt.*

For the remainder of the analysis, all statements refer to mean-field approximations.

4 Default Probability

4.1 From Individual to Systemic Default

By definition of leverage in (5), when banks have portfolios of largely overlapping external assets, leverage is positively correlated for all pairs of banks, i.e., $\text{Corr}(\phi_i, \phi_j) > 0 \quad \forall (i, j)$. Under this assumption, banks engage in similar activities (e.g., mortgages concentrated in specific areas, loans to specific sectors of the economy) and doing so, are, indirectly, linked as they are exposed to the same risk drivers. Therefore, large losses due to exogenous factors hitting a single bank, lead to a chain reaction in the financial network such that when a single bank is close to default, it is likely that several other banks are also close to default. If, at the same time, the financial network of claims and obligations is dense enough, the default of a single bank is very likely to have a systemic impact.

Let us define $\mathbb{P}'(\cdot)$ and $\mathbb{P}(\cdot)$ as the default probability of a single bank and the system, respectively. Moreover let the density of the network $\mathbf{W}_{n \times n}$ be the fraction between the number ℓ of actual links that exist between banks and the number of potential links that exist if all banks were connected through relationship links, i.e., $d_{\mathbf{W}} := \ell / n(n-1)$. Then, the following *Assumption* enables one to assign a systemic meaning to the results of the next Sections.

Assumption 4.1 *Let assume (i) banks manage overlapping portfolios s.t. $\text{Corr}(\phi_i, \phi_j) > 0$ for all $n - 1$ (i, j) pairs of banks in the system, (ii) the initial leverage ϕ_0 is homogeneous enough across banks, (iii) $\mathbf{W}_{n \times n}$ is sufficiently dense. Then, there exists $\epsilon > 0$ and $\ell^* \mid \forall \ell > \ell^*$*

$$| \mathbb{P}'(\cdot) - \mathbb{P}(\cdot) | < \epsilon .$$

The similarity in exposures toward external assets carries the potential for systemic breakdowns.[11] This potential is either strong or weak, depending on whether the density $d_{\mathbf{W}}$ remains or vanishes asymp-

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[10]Namely, a positive $\mathbb{E}(y_j)_t$ defines an uptrend. While, a negative $\mathbb{E}(y_j)_t$ maps a downtrend.

[11] According to the empirical evidence (see e.g., Elsinger et al., 2006; Iori et al., 2006), bank networks feature a core-periphery structure with a dense core of large banks and a periphery of small banks. Our hypothesis about the density of $\mathbf{W}_{n \times n}$ is thus realistic for the banks in the core.

totically. For the remainder, unless specified differently, the analysis will refer to the systemic default probability $\mathbb{P}(\cdot)$, rather than to the failure probability of a single bank.

4.2 Diversification and Default Probability

Under the conditions posed in *Assumption 4.1*, the leverage defined in (5), takes the meaning of *systemic fragility*. When a large part of banks manages highly leveraged balance-sheets, the system equilibrium is more sensitive to external shocks, such that even a small shock may trigger the system towards a distress regime. [12]

By definition, the process $\{\phi\}_{t \geq 0}$ driving the dynamics of the leverage is bounded between the lower *safe barrier* ($a_\phi := 0$) and the upper *default barrier* ($b_\phi := 1$). Formally, at any time $t > 0$, $\Omega_\phi = \{\phi \in \mathbb{R}^+ \mid a_\phi \leq \phi \leq b_\phi,\}$ represents the set of values taken by $\{\phi\}_{t \geq 0}$. Then, the default probability is defined as follows.

Proposition 2. *Default Probability (1). Consider the probability space $(\Omega_\phi, \mathcal{A}, \mathbb{P})$ with (i) $\mathbb{P}(\Omega) = 1$, (ii) $\mathbb{P}(\phi = a_\phi) + \mathbb{P}(\phi = b_\phi) = 1$, (iii) $\{\mathbb{P}(\phi = a_\phi) \in \mathbb{R} \mid 0 \leq \mathbb{P}(\phi = a_\phi) \leq 1\}$ and $\{\mathbb{P}(\phi = b_\phi) \in \mathbb{R} \mid 0 \leq \mathbb{P}(\phi = b_\phi) \leq 1\}$, the default probability of the financial system is the probability $\mathbb{P}(\phi = b_\phi)$ that the process $\{\phi\}_{t \geq 0}$, initially starting at an arbitrary level $\phi_0 \in (a_\phi, b_\phi)$ exits through the default barrier b_ϕ.*

Proof. See Appendix A. □

The leverage in (8) is directly influenced by the *non-financial factor* i.e., the cash-flows generated by investments in external assets in (16). Therefore, *Proposition 2*, based on the dynamics of the process $\{\phi\}_{t \geq 0}$, can be equivalently stated in terms of the underlying process $\{y\}_{t \geq 0}$.

Proposition 3. *Default Probability (2). Consider the sample space $\Omega_y = \{y \in \mathbb{R}^+ \mid a_y \leq y \leq b_y\}$, let $f^- : \Omega_\phi \to \Omega_y$ be a measurable function defined by the inverse of (8) that associates each element ϕ of Ω_ϕ with an element y of Ω_y, s.t., $y \in f^-(\phi)$. Then, the probability space $(\Omega_\phi, \mathcal{A}, \mathbb{P})$ in Proposition 1, can be mapped into the probability space $(\Omega_y, \mathcal{A}, \mathbb{P})$, s.t. the default probability of the financial system is the probability*

$$\mathbb{P}(y = a_y) = \left(\exp\left(-\frac{2\mu_y y_0}{\sigma_y^2}\right) - \exp\left(-\frac{2\mu_y b_y}{\sigma_y^2}\right)\right) \Big/ \left(\exp\left(-\frac{2\mu_y a_y}{\sigma_y^2}\right) - \exp\left(-\frac{2\mu_y b_y}{\sigma_y^2}\right)\right), \qquad (9)$$

that the process $\{y\}_{t \geq 0}$ in Proposition 1, initially starting at an arbitrary level $y_0 \in (a_y, b_y)$, exits through the default barrier $a_y := \bar{p}(r_f + \beta)/(R + \beta)$ with $\sigma_y := \sqrt{\frac{\sigma^2}{m} + \frac{m-1}{m}\bar{\rho}\sigma^2}$, $b_y = \bar{p}/2R$ and $y_0 \in (a_y, b_y)$.

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[12]Moreover, a status of system fragility is usually accompanied by costs (e.g., higher passive interest rates) which exacerbate the situation of the investors and may even lead to the collapse of the system.

Proof. See Appendix A. □

In the remainder, the analysis refers to *default probability (2)*. In particular, it is interesting to observe how $\mathbb{P}(y = a_y)$ is influenced by : (i) the number m of external assets or projects into which the banking system invests its wealth; (ii) the condition of the non-financial sectors, measured by the instantaneous aggregated expected growth rate μ_y.

Proposition 3, exactly maps together the value of the external assets with the extrema (maximum and minimum) of banking system fragility. The maximum level of financial system robustness – up to its maximum resilience –, occurs when $\phi = 0$ or equivalently when $y = \bar{p}2R$. On the contrary, the collapse of the banking system occurs when $\phi = 1$ or when $\bar{p}(r_f + \beta)/(R + \beta)$. In particular, an asymptotic analysis of (9), reveals that for increasing levels of diversification (namely, $m \to \infty$), $\mathbb{P}(y = a_y)$ exhibits a bifurcated behavior which depends on the condition of the non-financial sector underlying the external assets. Precisely, $\mathbb{P}(y = a_y)$ decreases with diversification in periods of upturn. While, it increases with diversification in periods of downturn. Then, the asymptotic behavior of $\mathbb{P}(y = a_y)$, yields the following *Proposition*

Proposition 4. *Banking diversification in external assets is desirable in upturn and is undesirable in downturn. The degree of (un)desirability increases with the level of diversification.*

Proof. See Appendix A. □

In words, *Proposition 4* says that the benefit of diversification in external assets is influenced by the returns generated by these activities. Essentially, In periods of upturn, banking diversification is desirable since the probability of default will asymptotically be reduced to zero as $m \to \infty$. While, in period of downturn, banking diversification is undesirable since the probability of default will asymptotically goes to one as $m \to \infty$. (See Figure 1 in Appendix). The intuition behind this polarization of probability to "survive" and probability to "fail" is beguilingly simple, but its implications profound. In upturn periods, the system acts as a mutual insurance device with disturbances dispersed and dissipated. Connectivity engenders robustness. In this case, diversification would serves as a shock-absorber. But in downturn, the system flips the wrong side of the knife-edge. Diversification serve as shock-amplifiers, not dampener, as losses cascade.

4.3   Optimal Diversification Policy

By *Proposition 3*, the payoff from investment is external assets, is bounded.

Definition 4.1. *Max Profit/Loss. The profit of the financial system is bounded between an upper bound at $y = b_y$ and a lower bound at $y = a_y$. Then, given the initial aggregate value of external assets $y_{t=0} := y_0$, the gain from investment in external assets is $\pi^+ := b_y - y_0$; while, the loss is $\pi^- := y_0 - a_y$.*

Based on the management strategy in *Definition 2.1*, banks may choose from a finite set $\mathcal{M}$ of diversification strategies m, with $m = 1, 2, \dots$. It represents a set of mutually exclusive alternatives s.t. only one of them may be chosen in any period Δt. The "buy and hold" asset allocation would be the following one.

Definition 4.2. *Asset Allocation*. *Select the number m of external investments (assets and/or projects) in which to invest an equal dollar amount. Then, hold the selected assets for an arbitrary period Δt.*

The payoffs resulting from the choice of a diversification strategy m are represented by a random variable Π_m that takes values π in the set $\Omega_\pi = [\pi^-, \dots, \pi^+]$. More specifically, given m alternatives $1, 2, \dots$ and their corresponding random return $\Pi_1, \Pi_2, \dots$, with distribution function $F_1(\pi), F_2(\pi), \dots$, respectively, preferences satisfying the von Neumann-Morgenstern axioms imply the existence of a measurable, continuous utility function $U(\pi)$ such that Π_1 is preferred to Π_2 if and only if $\mathbb{E}U(\Pi_1) > \mathbb{E}U(\Pi_2)$. We assume banks are mean-variance decision makers, such that the utility function $\mathbb{E}U(\Pi_m)$ may be written as a smooth function $V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right)$[13] of the mean $\mathbb{E}(\Pi_m)$ and the variance $\sigma^2(\Pi_m)$ of Π_m or $V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right) := \mathbb{E}U(\Pi_m) = \mathbb{E}(\Pi_m) - (\lambda\sigma^2(\Pi_m))/2$ so that Π_1 is preferred to Π_2 if and only if $V\left(\mathbb{E}(\Pi_1), \sigma^2(\Pi_1)\right) > V\left(\mathbb{E}(\Pi_2), \sigma^2(\Pi_2)\right)$. [14] Then, the maximization problem is

$$\max_m \mathbb{E}U(\Pi_m) = \mathbb{E}(\Pi_m) - \frac{\lambda\sigma^2(\Pi_m)}{2} \tag{10}$$

$$\text{s.t.:} \quad m \geq 1 \; ; \frac{1}{n} \sum_i^n \sum_j^m z_{ij} = 1 \; .$$

To simplify notations, let $p := \mathbb{P}(\mu_y \leq 0)$; $q := \mathbb{P}(y = a_y \mid \mu_y < 0)$; $g := \mathbb{P}(y = a_y \mid \mu_y > 0)$. Then, the expected profit $\mathbb{E}(\Pi_m)$ of the financial system is

$$\mathbb{E}(\Pi_m) = p\left[q\pi^- + (1-q)\pi^+\right] + (1-p)\left[g\pi^- + (1-g)\pi^+\right] \; . \tag{11}$$

In Table 1 in Appendix, the expected profit from banking diversification is considered in three different states of nature. (See Table 1 in Appendix). The variance $\sigma^2(\Pi_m)$ of the profit is

$$\sigma^2(\Pi_m) = p\left[q\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-q)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right]$$
$$+ (1-p)\left[g\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-g)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right] \; . \tag{12}$$

Here below we conduct an analytical study of $\mathbb{E}U(\Pi_m)$ for different values of parameters reported the Table 2 in Appendix.

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[13]To say that V is smooth it is simply meant that V is a twice differentiable function of the parameters $\mathbb{E}(\Pi_m)$ and $\sigma^2(\Pi_m)$.

[14]Only the first two moments matter for the decision maker, so the expected utility can be written as a function in terms of the expected return (increasing) and the variance (decreasing) only, with $\partial V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right)/\partial\mathbb{E}(\Pi_m) > 0$ and $\partial V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right)/\partial \sigma^2(\Pi_m) < 0$.

Analysis (i). In this analysis (10) is simplified since two mutually exclusive events are considered. First, the event downturn which occurs with probability $p = 1$. Second, the upturn event which occurs with probability $1 - p = 1$. In the downturn case, $\mathbb{E}U(\Pi_m)$ is maximized for decreasing range of diversification. While, in the upturn case, $\mathbb{E}U(\Pi_m)$ is maximized for increasing levels of diversification. Then, the following holds true.

Corollary 1. *Let m_{max} (m_{min}) be the max (min) attainable diversification strategy. Assume banks: (i) invest in external assets with dynamics defined in Proposition 1; (ii) use an allocation strategy as defined in 4.2; (iii) adopt a quadratic utility function of the form in (10). Then, in downturn, $m_{min} > m_{max}$. Conversely, in upturn, $m_{max} > m_{min}$.*

Proof. See Appendix B. □

The *Corollary 1* follows immediately from *Proposition 4*. When external assets pay (with probability one) a negative or positive cash-flow, banks' utility function in (10), respectively decreases or increases with increasing diversification. See results in Table 1, (Case 1 and Case 3) and see Figure 1 in Appendix.

Analysis (ii). In this analysis (10) is maximized w.r.t. m in the case the downturn and the upturn are two likely events that might occur with some probability. Then, the following Proposition holds.

Corollary 2. *Under the results in Corollary 1, let the event downturn (upturn) occur with probability p, $(1 - p)$ where $p \in \Omega_P := [0, 1]$. Then, there exists a subset $\Omega_{P*} \subset \Omega_P$ s.t., to each $p^* \in \Omega_{P*}$, corresponds an optimal strategy $m^* \in (m_{min}, m_{max})$ that maximizes the MV utility function in (10).*

Proof. See Appendix A. □

In words, let the absolute value of the trend to be given. Then, we assign a probability p to the event downturn and $1 - p$ to the upturn. Then, the expected utility $\mathbb{E}U(\Pi_m)$ is computed for given q and g. We found that for certain values of p^*, there exist a specific optimal level of diversification m^* which is a function of p^*. (See Figure 2 in appendix). *Corollary 2* suggests that only m^* is the optimal diversification strategy that has to be chosen in order to maximize the expected MV utility. Values of $m \gtrless m^*$ are second-best choices. In particular, for m approaching m^* from below, the system increases its robustness, while for m bigger then m^*, the system is beyond its *tipping point* and becomes more fragile. Then the utility function may exhibit a non-monotonic behavior with respect to m. (See Figure 2 in Appendix).

5   Private Incentives vs Social Welfare

Here we discuss the different implications of diversification at system level from a standpoint of a regulator. Let's consider the case of a policy maker that has to include some negative externalities which

might be engendered by losses occurred in downturn periods. Indeed, in an deeply downward trend, the wealth generated by the financial system is below the average-trend and might even be negative due to generalized losses and failures. Because of deadweight costs of systemic failure that exceed the costs of individual failures, the regulator it is plausibly to consider social costs[15] that might emerge due to the losses suffered by the financial system. [16] Then, if K is the number of simultaneously crashing banks, treating the policy maker as an expected utility maximizer, it is reasonable to assume the followings.

Assumption 5.1 *The total loss to be accounted by the regulator in downturn periods, is a monotonically increasing function $f(\cdot)$ of (i) the expected number k of bank crashes given a collapse of at least one bank $\mathbb{E}(K|K \geq 1) = k$, (ii) the magnitude of the loss π^-. Let the function $f(\cdot)$ be defined by $f(k, \pi^-) := k\pi^-$.*

Under this new perspective, the optimal diversification strategy (m^R) from the policy maker point of view does not coincide with the diversification desirable from the financial system point of view (m^*). More precisely,

Corollary 3. *Individual banks' incentives favor a financial network that is over-diversified in external assets w.r.t. to the level of diversification that is socially desirable*

$$m^* \geq m^R \ . \tag{13}$$

Proof. See Appendix A. □

In the presence of limited liability, the expected loss of an individual bank is bounded. In contrast, at the system level, either due to direct or indirect exposures, the externalities associated with the failure of interconnected institutions under *Assumption 4.1*, amplify the expected losses. These reflect the incremental costs due to a system collapse which are ultimately borne by taxpayers. Under this condition, *Corollary 3* says that the optimal level of diversification for the regulator is always smaller than the one desirable at individual level.

6   Concluding Remarks

In this paper, we have combined the balance-sheet approach (Shin, 2008) with a stochastic setting *a' la* Merton (1974). We consider a financial network of risk-averse banks, whose wealth is partially invested in assets external to the financial network, such as mortgages, loans to firms and other activities/projects related to the real-side of the economy. Starting from the law of motion of the value of assets and liabilities, we developed a parsimonious, yet micro-founded, stochastic framework in which the fragility of a bank depends on the fragility of the other banks and on the value of its external assets. In a mean-field

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[15]Social costs include costs of financial distress and economic distress.

[16]The definition of social losses is rather flexible since it depends on the characteristics of the financial system under analysis.

approximation, and under the assumption that banks choose the optimal number of external assets in a equally weighted portfolio, the failure probability can be derived analytically.

In this setting, we shed light on the conflict between one one hand the individual incentive to reduce, through diversification, the idiosyncratic risks of the external assets and, on the other hand, the emergence of systemic risk. In contrast with some optimistic views about diversification, but in line with other recent works (see e.g., Battiston et al., 2009; Ibragimov et al., 2011; Stiglitz, 2010; Wagner, 2009), we find that diversification fuels the *double-edged* property of the financial system. Indeed, diversification increases the default probability of the banking system in case external assets pay a negative cash-flow (downturn) and decreases the default probability in case of positive cash-flows (upturn).

Two main effects emerge in our model. The first is that the gap between maximal and minimal MV utility is exacerbated with higher degrees of diversification. Moreover, for a given fixed degree, the gap increases the larger is the absolute value of the growth rate (i.e., higher the divergence of returns between the up and down trend. This implies that, for a given probability of occurrence of the down(up) turn, there exists an optimal level of risk diversification which maximize the banking MV utility function.

By including in the analysis social costs due to generalized losses or defaults in case of a downturn, we obtain a final result. Individual banks' incentives favor a financial network that is over-diversified in external assets w.r.t. to the level of diversification that is socially desirable. Thus, to preserve a stable financial system, diversification should be encouraged in periods of economic boom and constrained in periods of recession. Nevertheless, as diversification of the financial system towards external activities is assumed to be a rigid strategy, the policy should be to adjust the system within a certain diversification range. In line with Ibragimov et al. (2011), our model suggests that policy restrictions to risk-sharing in upturn periods would limit excessive risk-spreading in downturn periods. Hence, an important point of our analysis is in recognizing that the objective of the regulator is not to target a specific level of default risk, but rather to manage the tradeoff between the social losses from defaults and the social costs of avoiding defaults.

A   Proofs

PROOF OF LEMMA 2.1. The leverage at banking system level can be described starting from (7) which can be rewritten as

$$\bar{p}_i = \phi_i \left(\sum_j^m \mathbf{Z}_{ij} y_j + \sum_j^n \mathbf{W}_{ij} \bar{p}_j / \left(1 + r_f + \beta \phi_j \right) \right)$$

which at system dimension, becomes

$$\bar{\mathbf{p}} = \mathbf{I}\phi \times \left[\mathbf{Zy} + (\mathbf{R} + \beta \mathbf{I}\phi)^{-1} \mathbf{W}\bar{\mathbf{p}} \right]$$
$$= \mathbf{\Phi} \times \left[\mathbf{Zy} + (\mathbf{R} + \beta \mathbf{\Phi})^{-1} \mathbf{W}\bar{\mathbf{p}} \right]$$
$$= \mathbf{\Phi Zy} + \mathbf{\Phi}(\mathbf{R} + \beta \mathbf{\Phi})^{-1} \mathbf{W}\bar{\mathbf{p}}.$$

Let define the column vector $\delta := \mathbf{W}\bar{\mathbf{p}}$. Then,

$$\bar{\mathbf{p}} = \mathbf{\Phi Zy} + \mathbf{\Phi}(\mathbf{R} + \beta \mathbf{\Phi})^{-1} \delta$$
$$\bar{\mathbf{p}}\delta^{\mathrm{T}} = \mathbf{\Phi Zy}\delta^{\mathrm{T}} + \mathbf{\Phi}(\mathbf{R} + \beta \mathbf{\Phi})^{-1} \delta\delta^{\mathrm{T}}$$
$$\bar{\mathbf{p}}\delta^{\mathrm{T}}(\delta\delta^{\mathrm{T}})^{-1} = \mathbf{\Phi Zy}\delta^{\mathrm{T}}(\delta\delta^{\mathrm{T}})^{-1} + \mathbf{\Phi}(\mathbf{R} + \beta \mathbf{\Phi})^{-1}$$
$$\bar{\mathbf{p}}\delta^{\mathrm{T}}(\delta\delta^{\mathrm{T}})^{-1}(\mathbf{R} + \beta \mathbf{\Phi}) = \mathbf{\Phi Zy}\delta^{\mathrm{T}}(\delta\delta^{\mathrm{T}})^{-1}(\mathbf{R} + \beta \mathbf{\Phi}) + \mathbf{\Phi}.$$

Let $\Delta := (\delta\delta^{\mathrm{T}})^{-1}$ and let $\mathbf{K} := \mathbf{Zy}\delta^{\mathrm{T}}\Delta$. Then,

$$\bar{\mathbf{p}}\delta^{\mathrm{T}}\Delta\mathbf{R} + \bar{\mathbf{p}}\delta^{\mathrm{T}}\Delta\beta\mathbf{\Phi} = \mathbf{\Phi KR} + \mathbf{\Phi K}\beta\mathbf{\Phi} + \mathbf{\Phi}$$
$$0 = \mathbf{\Phi K}\left[\mathbf{R} + \beta\mathbf{\Phi}\right] - \bar{\mathbf{p}}\delta^{\mathrm{T}}\Delta\beta\mathbf{\Phi} + \mathbf{\Phi} - \bar{\mathbf{p}}\delta^{\mathrm{T}}\Delta\mathbf{R}$$
$$= \mathbf{\Phi K}\left[\beta\mathbf{\Phi} + \mathbf{R}\right] + \mathbf{\Phi}\left[\mathbf{I} - \bar{\mathbf{p}}\delta^{\mathrm{T}}\Delta\beta\right] - \bar{\mathbf{p}}\delta^{\mathrm{T}}\Delta\mathbf{R},$$

which is a quadratic expression in $\mathbf{\Phi}$. $\square$

PROOF OF PROPOSITION 1. The portfolio composed by the linear combination of m processes described in (4), each of them weighted by $1/m$, is

$$dy = \frac{1}{m}\sum_{j=1}^m \mu_j dt + \frac{1}{m}\sum_{i=j}^m \sigma_j dB_j = \mu_y dt + ... \tag{14}$$

To derive the diffusion term, it is convenient to think about a sum of m Gaussian correlated error terms weighted by $\frac{1}{m}$. Namely, $\frac{1}{m}\sum_{j=1}^m dB_j$ which is equivalent to write $\frac{1}{m}\sum_{j=1}^m \sqrt{dt}\xi_j$ with $\xi_j \sim N(0,1)$. Taking the variance yields

$$\sigma_y^2 = \frac{dt}{(m)^2}\sum_{j=1}^m Var(\xi_j)\frac{dt}{(m)^2}\sqrt{Var(\xi_j)Var(\xi_l)}\sum_{j=1}^m\sum_{j\neq l}^m \rho_{jl}$$
$$= \frac{dt}{m}\bar{\sigma}_\xi^2 + \frac{dt}{(m)^2}\sqrt{\sigma_{\xi j}^2\sigma_{\xi l}^2}\sum_{j=1}^m\sum_{l\neq i}^m \rho_{jl}, \tag{15}$$

where $\bar{\sigma}_\xi^2$ is the average variance among the m's $\sigma_{\xi_{j=1,...m}}^2$. Since returns are assumed to be i.i.d., $\sigma_{\xi_j}^2$ is constant, say σ^2 for all $j = 1, ..., m$. Then multiplying and dividing the second term of (15) by $m - 1$ and taking its square root, we get $\sigma_y := \sqrt{\frac{\sigma^2}{m} + \frac{m-1}{m}\bar{\rho}\sigma^2}$ where $\bar{\rho} = \sum_{j=1}^m \sum_{l \neq j}^m \frac{\rho_{jl}}{m(m-1)}$ and $\xi \sim N(0, 1)$. Hence (14) becomes

$$dy = \mu_y dt + \sigma_y dB \quad \forall i = 1, ..., n ,\tag{16}$$

where $dB \sim N(0, dt)$. $\qquad\square$

PROOF OF PROPOSITION 2. From Gardiner (1985), the probability that the leverage initially starting at an arbitrary level $\phi_0 \in (a_\phi, b_\phi)$ exits through the default barrier b_ϕ is

$$\mathbb{P}(\phi = b_\phi) := \left(\int_{a_\phi}^{\phi_0} d\phi' \psi(\phi')\right) / \left(\int_{a_\phi}^{b_\phi} dy \psi(\phi')\right) ,\tag{17}$$

where $\psi(\phi') = \exp\left(\int_0^{\phi'} -\frac{2\mu_\phi}{\sigma_\phi^2} d\phi\right)$. While $\mu_\phi = y\mu_y \frac{\partial f(y)}{\partial y} + \frac{1}{2}y^2\sigma_y^2 \frac{\partial^2 f(y)}{\partial y^2}$ and $\sigma_\phi = y\sigma_y \frac{\partial f(y)}{\partial y}$ are derived by Itos Lemma applied to the function $\phi = f(y)$ given in (8) and using the dynamics of the process $\{y\}_{t \geq 0}$ in (16). $\qquad\square$

PROOF OF PROPOSITION 3. First let observe that the partial derivative of ϕ w.r.t. y is negative. Thus, when y goes up (down), ϕ goes down (up). This means that when $\{\phi\}_{t \geq 0}$ touch the upper (lower) barrier, $\{y\}_{t \geq 0}$ touches the lower (upper) corresponding barrier. Given the one-to-one function $f : \Omega_y \to \Omega_\phi$ where $\phi = f(y)$ has the explicit form of (8); from $f^{-1} : \Omega_\phi \to \Omega_y$ where $y \in f^{-1}(\phi)$, we obtain the following mapping:

$$\begin{cases} f(y) = 1 := b_\phi & \text{iff } f^{-1}(\phi) = \bar{p}(r_f + \beta)/(R + \beta) := a_y \\ f(y) = 0 := a_\phi & \text{iff } f^{-1}(\phi) = \bar{p}/2R := b_y . \end{cases}$$

Under this mapping, the default probability of the financial system is the probability $\mathbb{P}(y = a_y)$ that $\{y\}_{t \geq 0}$ *exit through the default barrier* $\bar{p}(r_f + \beta)/(R + \beta) := a_y$ *instead of the safe barrier* $\bar{p}/2R := b_y$. Then, the default probability can be expressed in two equivalent measures, as $\mathbb{P}(\phi = b_\phi)$ or $\mathbb{P}(y = a_y)$. Where $\mathbb{P}(y = a_y)$ is the probability that the process $\{y\}_{t \geq 0}$ in (16), initially starting at an arbitrary level $y_0 \in (a_y, b_y)$, exits through the default barrier $a_y := \bar{p}(r_f + \beta)/(R + \beta)$. From Gardiner (1985), this probability has the explicit form

$$\mathbb{P}(y = a_y) := \left(\int_{y_0}^{b_y} dy \psi(y)\right) / \left(\int_{a_y}^{b_y} dy \psi(y)\right) ,$$

where $\psi(x) = \exp\left(\int_0^x -\frac{2\mu_y}{\sigma_y^2} dy\right)$. Then, $\mathbb{P}(y = a_y)$ has the closed form solution

$$\mathbb{P}(y = a_y) = \left(\exp\left(-\frac{2\mu_y y_0}{\sigma_y^2}\right) - \exp\left(-\frac{2\mu_y b}{\sigma_y^2}\right)\right) / \left(\exp\left(-\frac{2\mu_y a}{\sigma_y^2}\right) - \exp\left(-\frac{2\mu_y b}{\sigma_y^2}\right)\right) ,$$

where $\sigma_y := \sqrt{\frac{\sigma^2}{m} + \frac{m-1}{m}\bar{\rho}\sigma^2}$, $a_y = \bar{p}(r_f + \beta)/(R + \beta)$, $b_y = \bar{p}/2R$ and $y_0 \in (a_y, b_y)$. $\qquad\square$

PROOF OF PROPOSITION 4. Let us first provide an heuristic explanation of results in Proposition 3. Consider μ_y to be a random variable $\mu_y : \Omega_{\mu_y} \mapsto \mathbb{R}$ defined on the probability space $(\Omega_{\mu_y}, \mathcal{A}, \mathbb{P})$ where $\Omega_{\mu_y} = [-1, 1]$ with

distribution function $F : \mathbb{R} \to \mathbb{R}$ of μ_y as $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \mathbb{P}(\mu_y \leq x)$ [17] To simplify notations, let $p := \mathbb{P}(\mu_y \leq 0)$; $q := \mathbb{P}(y = a_y \mid \mu_y < 0)$; $g := \mathbb{P}(y = a_y \mid \mu_y > 0)$. Now, let consider the solution of (16) i.e., $y = y_0 + \mu_y + \sigma_y Bt$ and for simplicity, let $\bar{\rho} = y_0 = 0$. The limit of y for $m \to \infty$, yields a measurable function $y = \mu_y$. Thus, y is also a random variable on Ω_{μ_y}, since the composition of a measurable function is also measurable. Now let bound the variation of y into an arbitrary small subset $\Xi \subset \Omega_{\mu_y}$ such that $y \in [-\xi, \xi]$. Then, for $\mu_y \leq 1$, the probability of y to be equal to the upper bound is $\mathbb{P}(y = \xi \mid \mu_y \leq 1) = 1 - \epsilon$. While, for $\mu_y \geq -1$, the probability of y to be equal to the lower bound is $\mathbb{P}(y = -\xi \mid \mu_y \geq -1) = 1 - \epsilon$. We can now obtain a more formal asymptotic result if we substitute $-\xi$ and ξ with a_y and b_y respectively. Let rewrite (9) as $(M^{y_0} - M^{b_y})/(M^{a_y} - M^{b_y})$ where $M = \exp\left(\frac{-2\mu_y}{\sigma_y^2}\right)$. After some arrangements, $\mathbb{P}(y = a)$ becomes $(M^{y_0 - b_y})(M^{a_y - b_y} - 1) - 1/(M^{a_y - b_y} - 1)$. Taking the limit for $m \to \infty$ we get that $\mu_y > 0 \Rightarrow \mathbb{P}(y = a_y \mid \mu_y > 0) = 0 - 0 = 0$ and for $\mu_y < 0 \Rightarrow \mathbb{P}(y = a_y \mid \mu_y < 0) = 0 - (-1) = 1$. Then,

$$\forall \epsilon > 0, \quad \exists \quad \bar{m} > 1 \mid (q - g) > 1 - \epsilon \quad \forall m > \bar{m}$$

□

PROOF OF COROLLARY 1. Based on the "state of nature" of external-assets, Corollary 1 shows the polarization of the expected utility for increasing levels of diversification in those assets. The proof of Corollary 1 moves from the results in Proposition 4. Let first study $\mathbb{P}(y = a_y)$ in (9) as a function of m and for the sake of simplicity, let $\bar{\rho} = \beta = r_f = 0$ and let $\sigma = \bar{p} = 1$. [18]. Then (9) becomes

$$\mathbb{P}(y = a_y) = \left(\exp(-2\mu_y y_0 m) - \exp(-\mu_y m)\right) / \left(1 - \exp(-\mu_y m)\right). \tag{18}$$

When $\mu_y > 0$ ($\mu_y < 0$), (18) is negative (positive) for any parameters' value in the set of Table 2. Its partial derivative w.r.t. m is

$$\frac{\partial \mathbb{P}(y = a_y)}{\partial m} = \mu_y \exp\left(\mu_y m(1 - 2y_0)\right)\left(\exp(2\mu_y y_0 m) - 1 + 2y_0 - 2y_0 \exp(\mu_y m)\right) / \left(\exp(\mu_y m) - 1\right)^2,$$

which is equal to zero if $\mu_y = 0$, negative if $\mu_y > 0$ and positive if $\mu_y < 0$. Stated otherwise, from the definition of q and g, we have $\frac{\partial q}{\partial m} > 0$ and $\frac{\partial g}{\partial m} < 0$. Let now, study how the utility function changes w.r.t. m in two mutually exclusive cases. Namely, when the external assets are in upturn or downturn period for a certainty.

(1) The case $p = 0$ implies an upturn (i.e., $\mu_y > 0$) and the utility function in (10) is simplified as

$$\mathbb{E}U(\Pi_m)_{\mu_y>0} = (g\pi^- + (1-g)\pi^+) - \lambda \left(g\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-g)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right)/2 \tag{19}$$

Then,

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y>0}}{\partial m} = \left(\pi^- \frac{\partial g}{\partial m} - \pi^+ \frac{\partial g}{\partial m}\right) + \frac{\partial g}{\partial m}\frac{\lambda}{2}\left(\Delta_+^2 - \Delta_-^2\right),$$

where $\Delta_+^2 := \left(\pi^+ - \frac{\partial g}{\partial m}(\pi^- - \pi^+)\right)^2$ and $\Delta_-^2 := \left(\pi^- - \frac{\partial g}{\partial m}(\pi^- - \pi^+)\right)^2$. After some rearrangements

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y>0}}{\partial m} = g\left(\frac{\partial g}{\partial m}\right)(\pi^- - \pi^+) > 0, \tag{20}$$

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[17]With the usual properties: (i) $F(x) \geq 0$; (ii) $\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} \mathbb{P}(\mu_y \leq x) = \mathbb{P}(\mu_y < +\infty) = 1$; (iii) $\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \mathbb{P}(\mu_y \leq x) = \mathbb{P}(\mu_y < -\infty) = 0$.

[18]By *Proposition 2*, in this simplified case, $y_0 \in [0, 1/2]$

which is positive since $\frac{\partial g}{\partial m} < 0$ and $\pi^- - \pi^+ < 0$. Then we conclude that, for $p = 0$, $\mathbb{E}U(\Pi_m)$ is an increasing function of m.

(2) In the case $p = 1$, there exists a downturn (i.e., $\mu_y < 0$) and the utility function in (10) simplifies as

$$\mathbb{E}U(\Pi_m)_{\mu_y<0} = (q\pi^- + (1-q)\pi^+) - \lambda \left(q \left(\pi^- - \mathbb{E}(\Pi_m) \right)^2 + (1-q) \left(\pi^+ - \mathbb{E}(\Pi_m) \right)^2 \right)/2 \; . \tag{21}$$

Then,

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y<0}}{\partial m} = \left(\pi^- \frac{\partial q}{\partial m} - \pi^+ \frac{\partial q}{\partial m} \right) + \frac{\partial q}{\partial m} \frac{\lambda}{2} \left(\Phi_+^2 - \Phi_-^2 \right) \; ,$$

where $\Phi_+^2 := \left(\pi^+ - \frac{\partial q}{\partial m} (\pi^- - \pi^+) \right)^2$ and $\Phi_-^2 := \left(\pi^- - \frac{\partial q}{\partial m} (\pi^- - \pi^+) \right)^2$. After some rearrangements,

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y<0}}{\partial m} = q \left(\frac{\partial q}{\partial m} \right) (\pi^- - \pi^+) < 0 \; , \tag{22}$$

which is negative since $\frac{\partial q}{\partial m} > 0$ and $\pi^- - \pi^+ < 0$. Then we conclude that, for $p = 1$, $\mathbb{E}U(\Pi_m)$ is a decreasing function of m.

Summarizing, cases (1) and (2) lead to the following result

$$\begin{cases} \mu_y < 0 \Rightarrow \frac{\partial \mathbb{E}U(\Pi_m)}{\partial m} < 0 \\ \mu_y > 0 \Rightarrow \frac{\partial \mathbb{E}U(\Pi_m)}{\partial m} > 0 \end{cases}$$

$$\therefore \text{ if } \begin{cases} p = 0, & V \left(\mathbb{E}(\Pi), \sigma^2(\Pi) \right)_{m_{max}} > V \left(\mathbb{E}(\Pi), \sigma^2(\Pi) \right)_{m_{min}} \iff m_{max} \succ m_{min} \\ p = 1, & V \left(\mathbb{E}(\Pi), \sigma^2(\Pi) \right)_{m_{min}} > V \left(\mathbb{E}(\Pi), \sigma^2(\Pi) \right)_{m_{max}} \iff m_{min} \succ m_{max} \end{cases}$$

where m_{max} (m_{min}) are respectively the max (min) attainable diversification strategies and $\succ$ stands for preference over strategies with $x \succ y$ is read "x i preferred to y". $\qquad\square$

PROOF OF COROLLARY 2. The Corollary 2 shows that in the case the downturn and upturn events are both likely to occurs with certain probability, there might exists an optimal level of diversification m^* which maximizes the expected utility function $\mathbb{E}U(\Pi_m)$ in (10). To prove it, let's decompose (10) as follow

$$\mathbb{E}U(\Pi_m)_{\mu_y>0} = (1-p) \left[(g\pi^- + (1-g)\pi^+) - \frac{\lambda}{2} \left(g \left(\pi^- - \mathbb{E}(\Pi_m) \right)^2 + (1-g) \left(\pi^+ - \mathbb{E}(\Pi_m) \right)^2 \right) \right] \tag{23a}$$

$$\mathbb{E}U(\Pi_m)_{\mu_y<0} = p \left[(q\pi^- + (1-q)\pi^+) - \frac{\lambda}{2} \left(q \left(\pi^- - \mathbb{E}(\Pi_m) \right)^2 + (1-q) \left(\pi^+ - \mathbb{E}(\Pi_m) \right)^2 \right) \right] . \tag{23b}$$

Notice that (23a) and (23b) are respectively (19) and (21) used in *Corollary 1* to which a probability p and $1 - p$ to occur, has been assigned. In other terms, (10) can be interpreted as a linear combination of (19) and (21). Now, let observe that (23a) is increasing in m, while (23b) is decreasing in m

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y>0}}{\partial m} = (1-p)g \left(\frac{\partial g}{\partial m} \right) (\pi^- - \pi^+) > 0 \tag{24a}$$

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y<0}}{\partial m} = pq \left(\frac{\partial q}{\partial m} \right) (\pi^- - \pi^+) < 0 \tag{24b}$$

When the partial derivatives in (24a) and (24b) are equal to each other, the $\mathbb{E}U(\Pi_m)$ is maximized w.r.t. m. The condition to be verified is to find the probability p^* that makes the two equations to be equivalent

$$\text{FOC:} \quad (1-p)g\left(\frac{\partial g}{\partial m}\right)(\pi^- - \pi^+) = pq\left(\frac{\partial q}{\partial m}\right)(\pi^- - \pi^+).$$

Discarding the trivial solution $\mu_y = 0$, the condition is satisfied for all

$$p^* = 1/\left(1 + \frac{q}{g}\left(\frac{\partial q}{\partial g}\right)\right) \in \Omega_{P^*} \subset \Omega_P := [0,1] \tag{25}$$

with $q = q(m^*)$, $g = g(m^*)$. Then, in a more general form, (25) can be written as $p^* = f[g(m^*); q(m^*)]$. Since g, q and f are all one-to-one and hence invertible functions, for a fixed value of p^*, must exists an m^* such that

$$m^* = \left[(g^{-1}; q^{-1}) \circ f^{-1}\right](p^*) \Rightarrow \exists \quad \mathbb{E}U(\Pi_{m^*}) \geq \mathbb{E}U(\Pi_m) \quad \forall m \gtrless m^*.$$

$\square$

PROOF OF COROLLARY 3. From Assumption 5.1, the total loss accounted by the policy maker in downturn periods is an increasing function of the expected number k of bank crashes given a collapse of at least one bank and the magnitude of the loss i.e., $k\pi^-$. Then, the expected utility of the regulator $\mathbb{E}U_R(\Pi_m)$ is differently expressed w.r.t. the utility from the bank point of view $\mathbb{E}U(\Pi_m)$ i.e., $\mathbb{E}U_R(\Pi_m) \neq \mathbb{E}U_R(\Pi_m)$. In particular, the changes involve (11) and (12) which are reformulated as follow

$$\mathbb{E}_R(\Pi_m) = p\left[qk\pi^- + (1-q)\pi^+\right] + (1-p)\left[g\pi^- + (1-g)\pi^+\right]$$

$$\sigma_R^2(\Pi_m) = p\left[q\left(k\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-q)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right] + (1-p)\left[g\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-g)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right].$$

Let follow the same reasoning used in Corollary 2 and decompose $\mathbb{E}U_R(\Pi_m)$ as follow

$$\mathbb{E}U_R(\Pi_m)_{\mu_y>0} = (1-p)\left[(g\pi^- + (1-g)\pi^+) - \frac{\lambda}{2}\left(g\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-g)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right)\right] \tag{26a}$$

$$\mathbb{E}U_R(\Pi_m)_{\mu_y<0} = p\left[(qk\pi^- + (1-q)\pi^+) - \frac{\lambda}{2}\left(q\left(k\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-q)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right)\right]. \tag{26b}$$

Let observe that the partial derivative w.r.t. m of (26b) is steeper then the partial derivative w.r.t. m in (24b). In particular,

$$\frac{\partial \mathbb{E}U_R(\Pi_m)_{\mu_y<0}}{\partial m} = pq\left(\frac{\partial q}{\partial m}\right)(k\pi^- - \pi^+) < \frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y<0}}{\partial m} = pq\left(\frac{\partial q}{\partial m}\right)(\pi^- - \pi^+) < 0$$

While

$$\frac{\partial \mathbb{E}U_R(\Pi_m)_{\mu_y>0}}{\partial m} \equiv \frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y>0}}{\partial m} = (1-p)g\left(\frac{\partial g}{\partial m}\right)(\pi^- - \pi^+)$$

since (26a) is not affected by Proposition 5.1 and remains equivalent to (23a). The condition to be verified is to find the probability p^R that makes the two equations to be equivalent

$$\text{FOC:} \quad (1-p)g\left(\frac{\partial g}{\partial m}\right)(\pi^- - \pi^+) = pq\left(\frac{\partial q}{\partial m}\right)(k\pi^- - \pi^+).$$

Discarding the trivial solution $\mu_y = 0$, the condition is satisfied for all

$$p^R = 1 \Big/ \left(1 + \frac{q}{g}\left(\frac{\partial q}{\partial g}\right)\upsilon\right) \in \Omega_{P^R} \subset \Omega_P := [0,1] \tag{27}$$

with $q = q(m^R)$, $g = g(m^R)$ and $\upsilon = (k\pi^- - \pi^+)/(\pi^- - \pi^+)$. In a more general form, (27) can be written as $p^R = f\left(\upsilon[g(m^*); q(m^*)]\right)$. Since g, q, υ and f are all one-to-one and hence invertible functions, for a fixed value of p^R, must exists an m^R such that $m^R = \left[(g^{-1}; q^{-1}) \circ \upsilon^- \circ f^{-1}\right](p^R)$. Eq.(27) is a decreasing function w.r.t. υ which is a constant function bigger than one as $k > 1$. Then, $p^R < p^*$. This implies that the diversification level m^R which is optimal from the policy maker point of view is lower than the level m^* desirable by the banking system

$$m^R = \left[(g^{-1}; q^{-1}) \circ \upsilon^{-1} \circ f^{-1}\right](p^R) < m^* = \left[(g^{-1}; q^{-1}) \circ f^{-1}\right](p^*)$$

$\square$

B   Tables

B.1   Expected Profit from Banking Diversification

Expected profit from banking diversification in the real-sector				
	External-assets trend	p	$\mathbb{E}(\Pi_m)$	$\lim\limits_{m\to+\infty}\mathbb{E}(\Pi_m)$
Case 1	up	0	$g\pi^- + (1-g)\pi^+$	π^+
Case 2	stable	0.5	$\frac{(q+g)\pi^- + (2-q-g)\pi^+}{2}$	$\frac{\pi^+ + \pi^-}{2}$
Case 3	down	1	$q\pi^- + (1-q)\pi^+$	π^-

Table 1: Expected profit from banking diversification in case of up. stable and down trend of non-financial assets.

First, let remind (from *Proposition 3*), the asymptotic behavior of $\mathbb{P}(y = a_y)$ for negative and positive values of μ_y which gives $\lim\limits_{m\to+\infty} q = 1$ and $\lim\limits_{m\to+\infty} g = 0$. In Case 1, the expected profit of the banking system increases with diversification because $\lim\limits_{m\to+\infty} g = 0$. In Case 2, we implicitly assume that $\mu_y \sim N(0, 1)$ since $\mathbb{P}(\mu_y \leq 0) = F(0) = 0.5$. The expected profit asymptotically, for $m \to +\infty$, converges to $(\pi^+ + \pi^-)/2$. While for $m \to 1$, both q and g converge to one and the expected profit become $\mathbb{E}(\pi) = \pi^- + \pi^+$. In Case 3, the expected profit decreases with banking diversification because $\lim\limits_{m\to+\infty} q = 1$.

B.2   Range of values for Parameters

Parameter	Set of Values
m	$\{m \in \mathbb{N} : m \geq 1\}$
σ	$\{\sigma \in \mathbb{R} : 0 < \sigma < 1\}$
r_f	$\{r_f \in \mathbb{R} : 0 < r_f < 1\}$
μ_y	$\{\mu_y \in \mathbb{R} : -1 \leq \mu_y \leq 1\}$
$\bar{\rho}$	$\{\bar{\rho} \in \mathbb{R} : 0 \leq \bar{\rho} \leq 1\}$
b_y	$\{b \in \mathbb{R} : b_y > 0\}$
a_y	$\{a \in \mathbb{R} : 0 < a_y < b\}$

Table 2: Range of values for each expl. variable.
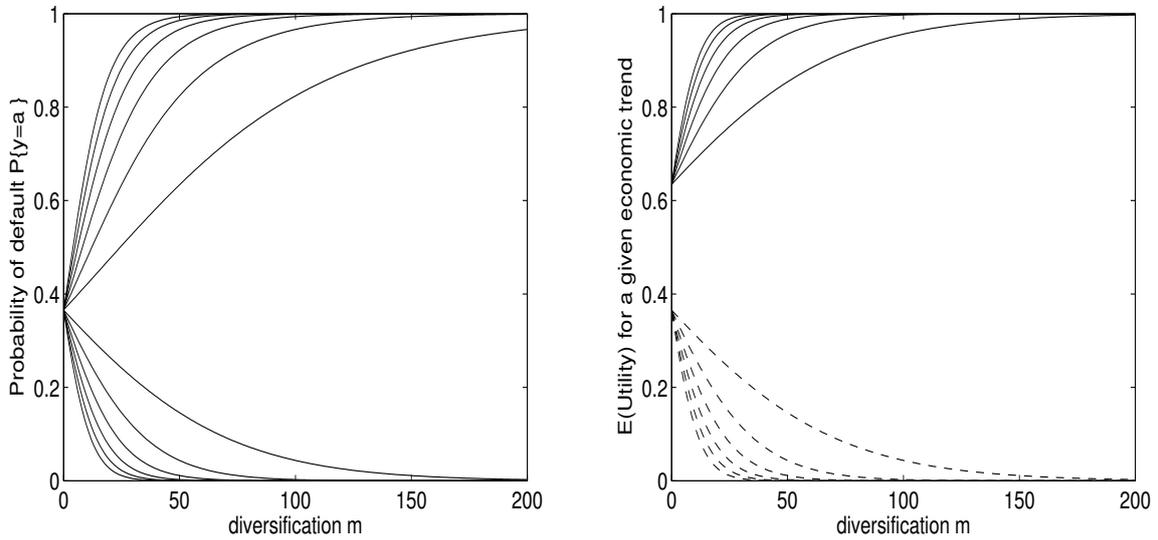
C   Figures

Figure 1: Default probability $\mathbb{P}(y = a_y)$ and Expected Utility $\mathbb{E}U(\Pi_m)$ of the banking system for different degrees m of diversification. *Left*: $\mathbb{P}(y = a_y)$ is decreasing in m when $\mu_y > 0$ (lower curves). $\mathbb{P}(y = a_y)$ is increasing in m when $\mu_y < 0$ (upper curves). *Right*: $\mathbb{E}U(\Pi_m)$ is decreasing in m when $\mu_y < 0$ (lower curves). $\mathbb{E}U(\Pi_m)$ is increasing in m when $\mu_y < 0$ (upper curves). Parameters: $\sigma^2 = 0.35$, $r_f = 0.03$, $\pi^- = \pi^+ = 1, \bar{\rho} = 0.1$, $m \in [0, 200]$, $|\mu_y| = 0.005, 0.01, 0.015, 0.020, 0.025, 0.030$.
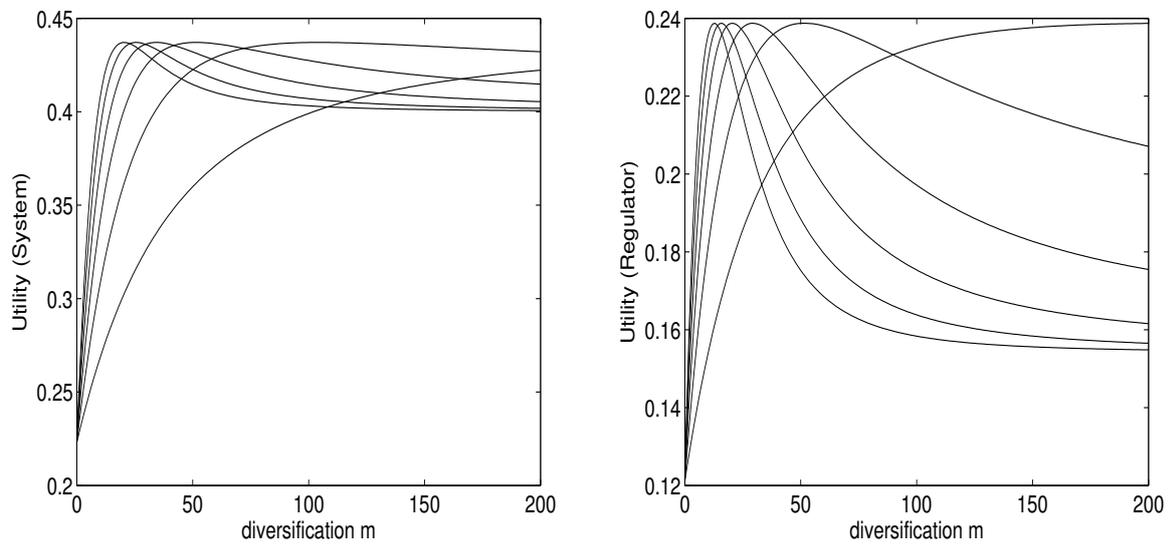
Figure 2: Expected Utility of the banking system $\mathbb{E}U(\Pi_m)$ (left) and Expected Utility of the regulator $\mathbb{E}U_R(\Pi_m)$ (right) for different degrees m of diversification. Parameters: $\sigma^2 = 0.35, r_f = 0.03,$ $\bar{\rho} = 0.1, p = 0.4, \pi^- = \pi^+ = 1, k = 2, m \in [0, 200], |\mu_y| = 0.005, 0.01, 0.015, 0.020, 0.025, 0.030$.

References

Acharya, V. (2009). A theory of systemic risk and design of prudential bank regulation. *Journal of Financial Stability*, 5(3):224–255.

Allen, F., Babus, A., and Carletti, E. (2010). Financial connections and systemic risk. Technical report, National Bureau of Economic Research.

Allen, F. and Gale, D. (2001). Financial contagion. *Journal of Political Economy*, 108(1):1–33.

Battiston, S., Gatti, D. D., Gallegati, M., Greenwald, B. C. N., and Stiglitz, J. E. (2009). Liaisons dangereuses: Increasing connectivity, risk sharing and systemic risk. *NBER Working Paper Series n.15611*.

Brock, W., Hommes, C., and Wagener, F. (2009). More hedging instruments may destabilize markets. *Journal of Economic Dynamics and Control*, 33(11):1912–1928.

Eisenberg, L. and Noe, T. (2001). Systemic risk in financial systems. *Management Science*, 47(2):236–249.

Elsinger, H., Lehar, A., and Summer, M. (2006). Risk Assessment for Banking Systems. *Management Science*, 52(9):1301–1314.

Gai, P. and Kapadia, S. (2007). Contagion in financial networks. *Bank of England, Working Paper*.

Gardiner, C. W. (1985). *Handbook of stochastic methods for physics, chemistry, and the natural sciences*. Springer.

Haldane, A. (2009). Rethinking the financial network. *Speech delivered at the Financial Student Association, Amsterdam, April*.

Haldane, A. and May, R. (2011). Systemic risk in banking ecosystems. *Nature*, 469(7330):351–355.

Ibragimov, R., Jaffee, D., and Walden, J. (2011). Diversification disasters. *Journal of Financial Economics*, 99(2):333–348.

Ibragimov, R. and Walden, J. (2007). The limits of diversification when losses may be large. *Journal of Banking and Finance*, 31(8):2551–2569.

Iori, G., Jafarey, S., and Padilla, F. (2006). Systemic risk on the interbank market. *Journal of Economic Behaviour and Organization*, 61(4):525–542.

May, R., Levin, S., and Sugihara, G. (2008). Complex systems: ecology for bankers. *Nature*, 451(7181):893–895.

Merton, R. (1974). On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *The Journal of Finance*, 29(2):449–470.

Nier, E., Yang, J., Yorulmazer, T., and Alentorn, A. (2007). Network models and financial stability. *Journal of Economic Dynamics and Control*, 31(6):2033–2060.

Shin, H. (2008). Risk and Liquidity in a System Context. *Journal of Financial Intermediation*, 17:315–329.

Shin, H. (2009). Securitisation and financial stability. *The Economic Journal*, 119(536):309–332.

Stiglitz, J. E. (2010). Risk and global economic architecture: Why full financial integration may be undesirable. *NBER Working Paper No. 15718*.

Trichet, J.-C. (2009). Systemic Risk. *Clare Distinguished Lecture in Economics and Public Policy*.

Vega-Redondo, F. (2007). *Complex Social Networks*. Series: Econometric Society Monographs. Cambridge University Press.

Wagner, W. (2009). Diversification at financial institutions and systemic crises. *Journal of Financial Intermediation*.