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Abstract

We disclose that only considering the largest cluster suffices to obtain a first-order percolation transition. As opposed to previous realizations of explosive percolation our novel models obtain Gaussian cluster distributions and compact clusters as one would expect at first-order transitions. We also discover that, surprisingly, the cluster perimeters are fractal at the transition point yielding a fractal dimension of 1.23 ± 0.03 intriguingly close to that of watersheds.

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Explosive percolation via control of the largest cluster

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We disclose that only considering the largest cluster suffices to obtain a first-order percolation transition. As opposed to previous realizations of explosive percolation our novel models obtain Gaussian cluster distributions and compact clusters as one would expect at first-order transitions. We also discover that, surprisingly, the cluster perimeters are fractal at the transition point yielding a fractal dimension of 1.23 ± 0.03 intriguingly close to that of watersheds.

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Percolation, the paradigm for random connectivity, has since Hammersley [1] been one of the most often applied statistical models [2, 3]. Its phase transition being related to magnetic models [4] is in all dimensions one of the most robust second-order transitions known. This explains the enormous excitement generated by the recent work by Achlioptas, D'Souza, and Spencer [5] describing a stochastic rule yielding a first-order percolation transition on a fully connected graph. Subsequent work applied the process on other networks [6–11]. Since then various rules have been devised [12–14] and even a Hamiltonian formalism was proposed [15], all resulting in a discontinuous transition towards an infinite cluster. In all proposed models one tries to keep the clusters of similar size and some authors additionally suppress the internal bonds of clusters [5, 15]. Evidently, these conditions are sufficient to produce the first-order transition but are they necessary? Could one obtain the same effect with less stringent rules? It is the objective of the present Letter to shed more light on this issue. Another open question about the Achlioptas process has been the cluster size distribution at the percolation threshold. Radicchi and Fortunato [9] as well as Ziff [7] found a power-law distribution with an exponent close to two. Although, different from the exponent of classical percolation the sole fact of finding a power law is untypical for first-order transitions. Also unusual for a first-order transition is that the clusters are fractal, as we found happens for the Achlioptas rule. It is a purpose of the present Letter to present a model in which a Gaussian cluster size distribution and compact clusters can be achieved in a systematic way, characterized by a fractal perimeter yielding a fractal dimension similar to the one of watersheds and random polymers in strongly disordered media.

Usual bond percolation can be implemented on a square lattice by randomly occupying bonds between neighboring sites, reaching its threshold at a certain fraction when opposite borders are first connected through one large cluster [2, 16, 17]. This percolation threshold, is characterized by the continuous vanishing of the order parameter, i.e., a second-order transition. On a fully connected graph, Achlioptas *et al.* [5], used the *best-of-two* product rule where, at each iteration, two new bonds are selected but only the one which connects the two clusters with the lowest product of their masses is occupied. This simple strategy is sufficient to change the behavior of the order parameter, being then characterized by a jump at the transition point, as studied in detail by Friedman and Landsberg [12]. Ziff reported that the product rule also induces a first-order transition on the regular square lattice [6, 7]. Independently, Radicchi and For-

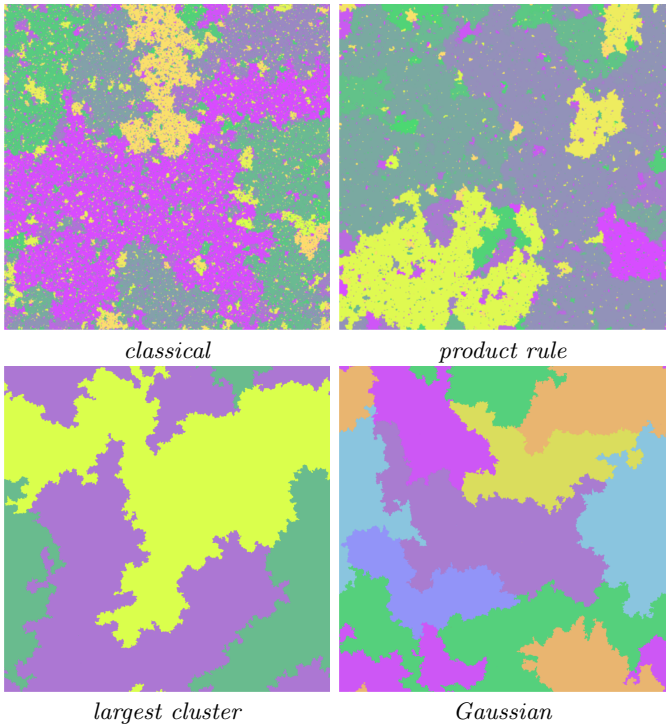

FIG. 1: (Color online) Snapshots of the system, obtained on a square lattice with 1024^2 sites, at p_c, for four different bond percolation models, namely, classical [2], Achlioptas product rule [6], *largest cluster* model ($\alpha = 1$), and *Gaussian* model ($\alpha = 1$). The *largest cluster* and *Gaussian* models are introduced in this Letter.

tunato [8, 9] and Cho *et al.* [10], have found the same transition when the Achlioptas process is considered on scale-free networks.

More recently, other approaches have been introduced to obtain explosive percolation. Instead of a *best-of-two* rule as explained before, Manna *et al.* [13], Cho *et al.* [11], and Moreira *et al.* [15] proposed a *weighted* rule where bonds are occupied according to a certain probability. However, despite being *rejection-free* schemes, they are limited to small-system sizes and/or reduced number of samples. Here, we suggest an acceptance method where new bonds are selected randomly and occupied according to a certain weight. The considered scheme allows to consider system sizes 64 times larger than before [13], specifically, we consider systems of 4096^2 sites and averages over 10^4 samples.

In our simplest rule ("*largest cluster* model"), as for classical bond percolation, a link is randomly selected among the empty ones. If its occupation would not lead to the formation or growth of the largest cluster, it is always occupied, otherwise, it is occupied with probability

$$\min\left\{1, \exp\left[-\alpha\left(\frac{s-\bar{s}}{\bar{s}}\right)^2\right]\right\} \quad , \tag{1}$$

where s is the size of the cluster that would be formed by occupying this bond and $\bar{s}$ the average cluster size after occupying the bond. The parameter α controls the allowed size dispersion. Note that, for $\alpha \leq 0$, since the size of the largest cluster is always greater (or equal) than the average cluster size, all new bonds are occupied reducing to classical bond percolation, characterized by a continuous transition at the percolation threshold [2]. For $\alpha > 0$, the probability of Eq. (1) suppresses the formation of a cluster significantly larger than the average, inducing a homogenization of cluster sizes.

For nonequilibrium problems, where a free energy cannot be defined, transitions can still be classified based on the behavior of the order parameter [18]. A first-order transition, is characterized by a jump in the order parameter, otherwise, a transition is denoted as continuous. For percolation, we define as order parameter the fraction of sites in the largest cluster (P_∞) [2]. Here we also consider two other quantities: the second moment of the cluster size distribution (χ), defined as

$$\chi = \sum_i s_i^2 \quad , \tag{2}$$

where the sum runs over all clusters i, and the standard deviation (χ_∞) of the largest cluster size (s_{max}) over different samples,

$$\chi_\infty = \sqrt{\langle s_{max}^2 \rangle - \langle s_{max} \rangle^2} \quad . \tag{3}$$

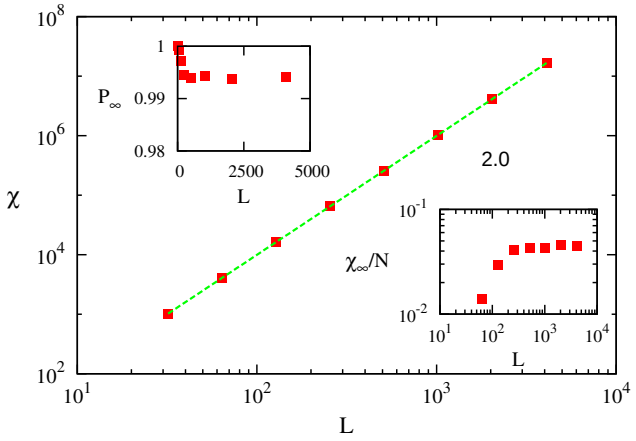

FIG. 2: (Color online) Size dependence, for the *largest cluster* model, of the susceptibility (χ), fraction of sites in the largest cluster (P_∞), and its standard deviation per site (χ_∞/N) at the percolation threshold, on a square lattice of linear size (L) ranging from 32 to 4096. All bonds are occupied with the same probability except the ones that lead to the formation/growth of the largest cluster, to which an occupation probability q is assigned, Eq. (1), with $\alpha = 1$. Results have been averaged over 10^4 samples.

To estimate the percolation threshold we consider the average value of p (fraction of occupied bonds) at which a connected path linking opposite boundaries of the system is obtained. Considering different system sizes, for $\alpha = 1$, we obtain for the percolation threshold $p_c = 0.632 \pm 0.002$. To identify the order of the transition, in the *largest cluster* model, Fig. 2 presents a finite-size study for P_∞, χ, and χ_∞/N, averaged over 10^4 samples of square lattices with linear sizes ranging from 32 to 4096. As we can see in the top inset of Fig. 2, above a certain system size, the order parameter, at the percolation threshold, does not show any finite-size dependence, staying at a constant value in the thermodynamic limit ($L \to \infty$). The second moment of the cluster size distribution (χ) scales with $L^d (d = 2)$ which is a sign of a first-order transition [19, 20]. The standard deviation of the largest cluster (s_{max}) per lattice site, which was also considered in Refs. [6] and [7], converges, for larger system sizes, to a constant value, corroborating the presence of a discontinuous transition.

Despite leading to a first-order transition, the Achlioptas process generates a power-law distribution of the cluster size [7, 9] (see Fig. 1). To explicitly control the cluster size distribution we also implemented the following model. A new bond is chosen from the list of empty ones and occupied with probability

$$\min\left\{1, \exp\left[-\alpha\left(\frac{s-\bar{s}}{\bar{s}}\right)^2\right]\right\} \quad , \tag{4}$$

where s is the size of the cluster obtained when the se-

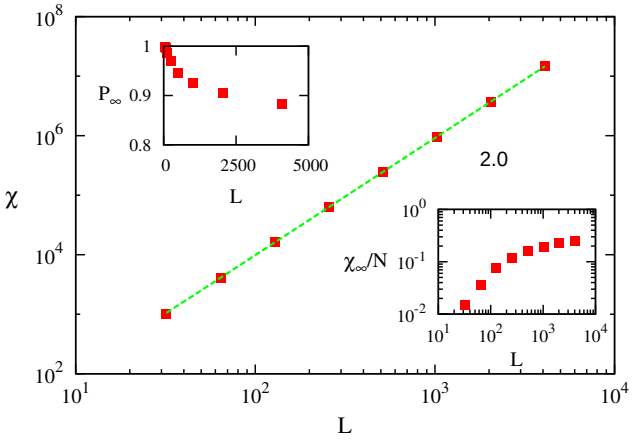

FIG. 3: (Color online) Size dependence, for the *Gaussian* model, with $\alpha = 1$, of the susceptibility (χ), fraction of sites in the largest cluster (P_∞), and its standard deviation per site (χ_∞/N) at the percolation threshold, on a square lattice of linear size (L) ranging from 32 to 4096. All bonds are occupied with a probability given by Eq. (4). Results have been averaged over 10^4 samples.

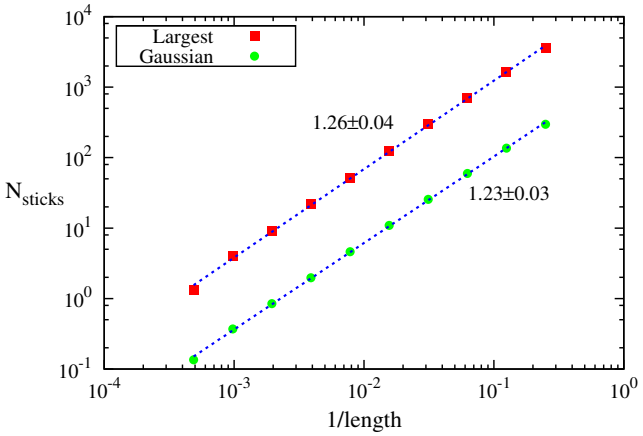

FIG. 4: (Color online) Number of sticks necessary to follow the perimeter of the infinite cluster as a function of the stick length, to obtain the fractal dimension of the perimeter with the *yardstick method*. For both the *largest cluster* and *Gaussian* models, with $\alpha = 1$. For the *Gaussian* model data were vertically shifted by a factor of 0.1. Results have been averaged over 10^4 samples of lattices with linear size 2048.

lected bond is occupied. For internal connections we consider s as twice the cluster size. Since equation (4) is a Gaussian with average size $\bar{s}$ and size dispersion $\bar{s}/\sqrt{2\alpha}$, we denote this model as *Gaussian* model. Note that here the occupation probability is assigned to all new bonds even when they are not related to the largest cluster. This not only guarantees the control over clusters greater than the average, as in the previous model, but also over the smaller ones. For $\alpha = 0$, all bonds have the same probability and, therefore, the model reduces to classical bond percolation. For negative α, the growth of larger clusters is favored in two different ways: they differ more from the average value and have more empty bonds than the smaller ones. Yet, for all negative α, the model recovers the classical universality class of percolation [2, 18].

As example, for positive α, we present, in Fig. 3, a size dependence study of the order parameter, second moment of the cluster size distribution, and standard deviation per site of the largest cluster, for the *Gaussian* model, with $\alpha = 1$, at the percolation threshold, on a regular square lattice with linear size (L) ranging from 32 to 4096. Results were averaged over 10^4 samples. We extrapolate, for the infinite system, a percolation threshold $p_c = 0.56244 \pm 0.00006$. As for the *largest cluster* model, the density of the infinite cluster does not change significantly with the system size, the second moment of the cluster size distribution scales with $L^d (d = 2)$, and the standard deviation per site of the largest cluster converges to a non-zero constant. As before these results imply a first-order transition.

Figure 1 shows snapshots for four different models of bond percolation: classical, product rule (Achlioptas process), *largest cluster* model, and *Gaussian* model. All figures have been obtained at their respective percolation thresholds (p_c). For classical percolation and for the Achlioptas process, clusters of very different sizes are obtained. In fact, the cluster size distribution is characterized by a power law [7, 9]. However, for the *largest cluster* and the *Gaussian* model, a characteristic cluster size is observed. Both models lead to a localized cluster size distribution. The smaller size dispersion and number of clusters are observed for the *largest cluster* model. According to Eq. (1), increasing the value of α decreases the size dispersion.

Clusters obtained with classical bond percolation and Achlioptas rule are fractal with holes inside. As clearly seen in the snapshots of Fig. 1, clusters obtained with our models are compact but astonishingly we find that the surface is fractal. For the *Gaussian* model, we calculate for the cluster perimeter a fractal dimension of 1.23±0.03, obtained with the *yardstick method* [21] (Fig. 4). For the *largest cluster* model, it is also characterized by a fractal perimeter with a fractal dimension of 1.26 ± 0.04 (Fig. 4). Compact clusters with fractal surface were also reported for irreversible aggregation growth in the limit of high concentration by Kolb *et al.* [22].

In Fig. 5 we see the cluster size distribution, $P(s, \alpha)$, for different system sizes, obtained with the *Gaussian* model. Measurements have been performed at the percolation threshold on a square lattice with 1024^2, 2048^2, and 4096^2 sites, and averaged over 10^4 samples. Three characteristic peaks are observed. In fact, the third peak (around 0.7) is only due to the largest cluster and only appears due to the small number of clusters at the per-

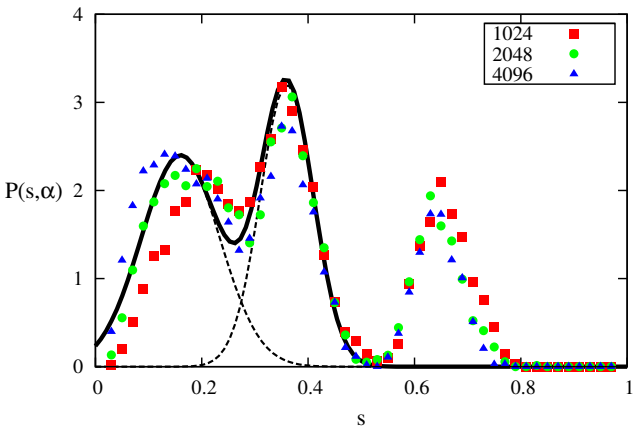

FIG. 5: (Color online) Cluster size distribution for the *Gaussian* model for different system sizes ($\alpha = 1$), at the percolation threshold, on a square lattice, averaged over 10^4 samples. Black-dashed lines are two Gaussian distributions fitting the results from simulation. The black-solid line is the sum of both curves.

colation threshold, being finite-size effect. This peak is not observed when we compute the same distribution neglecting the contribution of the largest cluster. The presence of two main peaks is characteristic for a first-order transition showing, for a finite system, at the percolation threshold, coexistence of the percolative and non-percolative states [23].

In conclusion, the present work reveals that, to obtain explosive percolation on a regular lattice it is sufficient to control the formation and growth of the largest cluster, instead of applying a rule to the overall set of empty bonds. We propose the *largest cluster* model which systematically suppresses the formation of a largest cluster. We introduce as well, the *Gaussian* model, where a weight is assigned to each selected bond, such that a Gaussian distribution of cluster sizes is obtained, revealing the coexistence of two states at the percolation threshold. Our clusters are compact instead of fractal as observed for the classical and product rules. However, curiously, for both models the perimeter of the largest cluster is fractal. Our models, yielding clear first-order transitions, show that explosive percolation can be obtained under much less stringent conditions that previously thought shedding light on the minimum ingredients to trigger explosive percolation. In fact, we believe that our restrictions on the formation of the largest cluster is the required necessary condition and hope that this statement can one day be formally proven. The novel fractal dimension that we discovered in the cluster perimeters is intriguingly close to the one found for watersheds [24] and random polymers in strongly disordered media [25], and we conjecture that it is identical.

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