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Abstract

"Disorder-induced volatility" (DIV) describes the enhanced fluctuations of collective behaviors exhibited by bistable systems in the presence of a rapidly fluctuating external signal. At the DIV resonance, a defining characteristics is that the response of the system becomes uncorrelated with the external driving noise, making DIV resonance different from stochastic resonance. Numerical simulations and an analytical theory of a stochastic dynamical version of the Ising model on regular and random networks demonstrate the ubiquity and robustness of DIV, which is proposed as a possible cause of excess volatility in financial markets, of enhanced effective temperatures in a variety of out-of-equilibrium systems and of strong selective responses of immune systems of complex biological organisms.

Keywords: volatility, stochastic processes, financial markets

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Notes and Comments:

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"Disorder-induced volatility" (DIV) describes the enhanced fluctuations of collective behaviors exhibited by bistable systems in the presence of a rapidly fluctuating external signal. At the DIV resonance, a defining characteristics is that the response of the system becomes uncorrelated with the external driving noise, making DIV resonance different from stochastic resonance. Numerical simulations and an analytical theory of a stochastic dynamical version of the Ising model on regular and random networks demonstrate the ubiquity and robustness of DIV, which is proposed as a possible cause of excess volatility in financial markets, of enhanced effective temperatures in a variety of out-of-equilibrium systems and of strong selective responses of immune systems of complex biological organisms.

Noise may have surprising effects on the organization of complex systems made of interacting elements, as shown by *stochastic resonance* (SR) [1], *coherence resonance* [2], *noise-induced phase transitions* [3], noise-induced transport [4] and its game theoretical version, the *Parrondo's Paradox* [5]. SR occurs in a system when a small applied (sub-threshold) periodic signal is amplified by noise of intermediate amplitudes. For instance in bistable systems, the optimal level of noise is such that the Kramers time equals half of the period of the external forcing. More generally, SR refers to the situation where noise and non-linearity combine to increase the order and strength in the system response. Among others, SR is thought to be relevant to optical and magnetic systems, to the Earth climate and the dynamics of ice ages, to neurobiology, and to medical instrumentations.

Research on SR has focused on slow periodic signals (of the order of the Kramers time), or on excitable systems [6], where the external (periodic or aperiodic) forcing has a time-scale comparable to that of the refractory time. Also, SR occurs in extended systems composed of many constituents, a paradigmatic example being the Ising model [7, 8]. The concept of SR was later extended, and it was shown that the exact source of disorder is irrelevant, as either noise, quenched disorder [9] or network heterogeneity [10] can cause a response enhancement.

Here, we report the existence of a new regime of strong amplification of the response of extended systems, which is characterized by two distinctly new features: (i) the system is driven by a rapidly varying noise (ii) the response of the system becomes uncorrelated from the forcing noise at the resonance. We document this "disorder-induced volatility" (DIV) by numerical and theoretical calculations on a stochastic dynamical version of the Ising model on regular and random networks in the presence of rapidly varying driving noise. DIV may help to understand the often surprising organization and paradoxes observed in complex systems in the presence of noise. A first example refers to the empirical observations of strong amplifications of thermal noise into effective renormalized temperatures by quenched heterogeneities in materials [11] in organized flows in liquids [12] and in granular media near jamming [13]. We suggest that DIV resonance also provides a conceptual framework to model the immune systems of complex biological organisms, viewed as multistable complexes, which switch their mode of operation under the influence of noisy perturbations by pathogens and other stress factors [14–16]. Another important application is the phenomenon of "excess volatility" [17], which constitutes the most blatant violation of the efficient market hypothesis of financial economics [18]. In a nutshell, excess volatility refers to the ubiquitous observations that financial prices fluctuate with much larger amplitudes than they should if they obeyed the fundamental valuation formula linking them to the firm dividends and discount factors [19]. The model described below can be applied to represent a market of interacting investors, which are subjected to a flow of news. In the sequel, we present the model and its result, in the language of three applications: effective enhanced temperature in material systems going to rupture, the immune system of complex biological organisms and financial markets.

Model. Consider a system made of N interacting units that can be in one of two states: $s = \pm 1$. The units are updated sequentially, randomly chosen at each unit micro-time $\delta = 1/N$, i.e. N updates are equivalent to one macro-unit time. The update from t to $t + \delta$ of the state s_i of a given unit i is given by

$$s_i(t + \delta) = \text{sign}\left(F(t) + \xi_i(t) + K(t) \sum_{j=1}^{N} \omega_{ij}\, s_j(t) \right) \quad (1)$$

If $F(t) = 0$ and $\xi_i(t)$ is distributed according to a Logistic distribution, this dynamical rule (1) is equivalent to the Ising model with Glauber dynamics.

Three contributions compete in deciding the value $s_i(t+\delta)$: (i) a common external forcing term $F(t)$ (force, pathogens, news); (ii) a unit-specific term $\xi_i(t)$, which can be annealed or quenched (thermal fluctuations or threshold, intrinsic susceptibility of a unit immune system compartment, investor idiosyncratic opinion or information); (iii) an interaction term between units controlled by the amplitude $K(t)$ (elastic coupling, feedback loops between immune system elements, social impact).

The external force $F(t)$ is assumed to change over a correlation time $1/\nu$. For simplicity, we take $F(t)$ equal to constants in plateaux of durations $\sim 1/\nu$. The plateau values f_n are i.i.d. and drawn from a probability distribution function (pdf) $f_n \sim D(0, A)$, with zero mean and variance A^2. The standard deviation A quantifies the strength of the external forcing. The change of plateaux can be periodic (with period $1/\nu$), Poisson (with intensity ν) or intermediate between these two limiting cases. The results presented here remain unchanged. We thus show the simplest case of a periodic update of $F(t)$ at every $\lfloor 1/\nu \rfloor$ micro-time steps (where $\lfloor 1/\nu \rfloor$ denotes the integer part of $1/\nu$):

$$F(t) = \sum_{n=0}^{\infty} \delta_{\lfloor t\nu \rfloor, n} f_n \ . \tag{2}$$

The covariance of this process (2) is given by

$$\langle\langle F(t + \theta) F(t) \rangle\rangle_t = \begin{cases} 1 - \nu|\theta| & \text{for } |\theta| < 1/\nu \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

where $\langle \rangle_t$ and $\langle \rangle$ denote respectively the temporal and ensemble averages.

In the annealed version, each idiosyncratic term $\xi_i(t)$ to each unit i in (1) follows an independent stochastic process, whose values are drawn from the cumulative distribution function $G(0, Q)$, with zero mean ($\langle \xi_i(t) \rangle = 0$) and variance Q^2, that we will call *disorder*. Thus, $\langle \xi_i(t) \xi_j(t') \rangle = Q^2 \delta(t - t') \delta_{ij}$.

In the interaction term in Eq. (1), the matrix of weights ω_{ij} defines the network between units, both in topology and relative strength. We assume that the feedbacks between units are governed by connections that evolve much slower than the dynamics of the whole system. This amounts to considering a static network with fixed normalized weights $\sum_j \omega_{ij} = 1$. The coupling strength of all other units on a given one is quantified by $K(t)$, which may depend on time to reflect global softening-hardening in rupture processes, evolving physiological states of immune systems and changes of social cohesiveness and/or propensity to imitate in financial markets.

The macroscopic dynamics of the system is captured by the instantaneous "magnetization" $r(t) =$
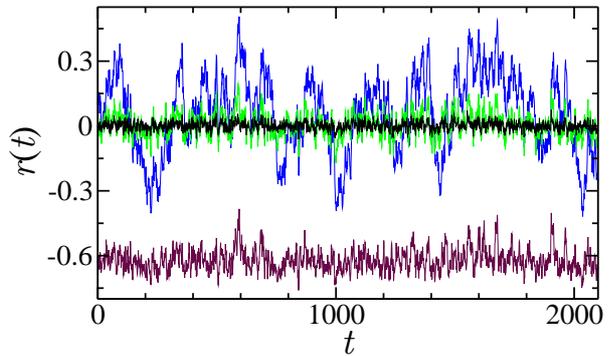
FIG. 1: (color online) Time evolution of the "magnetization" $r(t)$ for different disorder Q obtained with the same realization of the driving force $F(t)$ and $\xi_i(t)$. $N = 10^4$, $A = 0.04$, $k = 1.0$, $Q = 2.0, 1.0, 0.8$ (smaller to larger amplitude of $r(t)$'s fluctuations) and $Q = 0.7$ (bottom curve fluctuating around $r(t) = -0.6$).

$N^{-1} \sum_i s_i(t)$. We study the normalized standard deviation $\sigma_r/A = \langle (r(t) - \langle r(t) \rangle_t)^2 \rangle_t^{1/2}/A$ of the fluctuations of $r(t)$ in the time domain, referred to as the "volatility" of the response of the system in units of the external driving force of amplitude A. The cross-correlation between the input signal and the magnetization, defined by $\rho = \langle (r(t) - \langle r(t) \rangle_t) f(t) \rangle_t / A \sigma_r$, provides an additional insight on the level of synchronization between the external influence and the overall system dynamics.

We first consider complete homogeneous networks ($\omega_{ij} = 1/(N-1)$) and constant coupling strength $K(t) = k$. The results reported below are not significantly different for random graphs with large average connectivity or if the connections allow for an unbiased statistical sampling within the population. Fig. 1 illustrates the typical dynamic behaviors of $r(t)$ for different values of Q for a single realization of the driving force $F(t)$. A phase transition separates two regimes as a function of Q/k: for $Q > Q_c \simeq 0.80$ (for $k = 1$, and for a Gaussian distribution G), $r(t)$ fluctuates around 0 while, for $Q < Q_c$, it fluctuates around one of the two stable fixed points $\pm r_0(Q)$. The dependence of the time average level $r_0(Q)$ of $r(t)$ as a function of Q is shown in Fig. 2 and corresponds to the standard supercritical pitchfork bifurcation expected for an Ising-like system. Surprisingly, for an intermediate value of disorder ($Q = 0.8$), the fluctuations around the mean are much larger than for small or large disorder intensities.

Figure 2 reports, for three system sizes and three driving force amplitudes, the dependence of the two variables σ_r/A and ρ as a function disorder level Q obtained by numerical simulations of expression (1), which are compared with the analytical theory presented below. For large Q, the volatility is controled by the units with $\xi_i < F$, yielding $\sigma_r \sim Q^{-1}$, and leading to a large cross-correlation ρ. For $Q \approx Q_c$, the volatility amplifies several
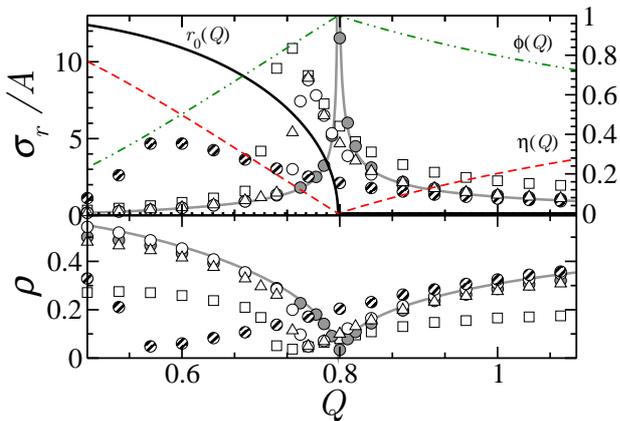
FIG. 2: Upper panel: (left scale) Scaled volatility σ_r/A as a function of Q; (right scale) the thick continuous (resp. dash-dotted and dashed) line is $r_0(Q)$ (resp. signal pre-factor $\phi(Q)$ and $\eta(Q)$ defined in (5)). Lower panel: cross-correlation ρ between $r(t)$ and $F(t)$ as a function of Q. Continuous lines: theoretical predictions (see text); symbols: numerical simulations of the model with $k = 1$. Open symbols: moderate driving force amplitude $A = 0.04$ with different system sizes $N = 900$ (squares), 10^4 (triangles) and 10^6 (circles). Filled circles: $A = 5 \times 10^{-3}$ (gray) and $A = 0.16$ (stripped) with $N = 10^6$.

times that of the driving signal $F(t)$. Concomitantly, ρ tends to vanish as the volatility of the system is generated by an internal collective behavior. For small A and Q significantly smaller than Q_c, the volatility σ_r/A becomes smaller than A, reflecting weak fluctuations of $r(t)$ associated with an almost saturated "ferromagnetic" state under weak forcing. We observe that larger external forcing values A recover an amplification of the volatility ($\sigma_r/A > 1$), which is associated with the jumps between $+r_0$ and $-r_0$ driven by the external force. Increasing A even further, such that the signal drives the system dynamics, leads to a slight monotonic decrease in volatility and an increase of cross-correlation. This large regime is similar SR, where supra-threshold signals are not amplified by the addition of noise.

The numerical results shown in Fig. 2 can be rationalized by the following mean-field theory. Averaging Eq. (1) over the population of units and taking the continuous limit, the dynamics reads

$$\frac{d}{dt}r(t) = -r(t) + 1 - 2G\left(-k\,r(t) - F(t)\right). \quad (4)$$

For $F = 0$ (no external driving), the stationary solution of (4) gives the dependence of the time averaged magnetization $r_0(Q)$ given by the solution of the implicit equation $r_0(Q) = 1 - 2G(-k\,r_0(Q))$. This solution, corresponding to a supercritical pitchfork bifurcation, is shown in Fig. 2. The critical parameter is found equal to $Q_c = k\sqrt{2/\pi}$, for $\xi_i(t)$ drawn from a Gaussian distribution.

For weak external forcing $A \ll 1$, a perturbation ex-

pansion $r(t) = r_0 + r_1(t)$ to linear order yields

$$\dot{r}_1(t) = \phi(Q)\,F(t) - \eta(Q)\,r_1(t)\,, \quad (5)$$

where $\phi(Q) \equiv 2g(-kr_0)$ and $\eta(Q) \equiv 1 - 2kg(-kr_0)$ are represented in Fig. 2 and $g = dG/d\xi$. If $F(t)$ was constant, $r_1(t)$ would tend at long times to $F\phi(Q)/\eta(Q)$. Since $\phi(Q)$ remains finite when Q passes through Q_c, it is the vanishing of $\eta(Q)$ at $Q = Q_c$ and its smallness in the vicinity of Q_c that is the origin of the amplified volatility shown in Fig. 2. For the time-dependent $F(t)$ described by expression (2), the exact solution of (5) reads $r_1(t) = \frac{\phi}{\eta}\left[(e^{\eta/\nu} - 1)\,e^{-\eta t}\sum + f_{\lfloor t\nu\rfloor+1}\left(1 - e^{-\eta(t-\lfloor t\nu\rfloor/\nu)}\right)\right]$, where $\sum = \sum_{i=0}^{\lfloor t\nu\rfloor-1} f_i\,e^{i\eta/\nu}$. The time-dependent perturbation $r_1(t)$ is thus the sum of terms linear in the stochastic variables f_i's. Using the fact that the plateau values f_i are i.i.d. and $f_i \sim D(0, A)$, the variance of r_1 obtained by time averaging and ensemble averaging over the random variables f_i is given by

$$\sigma_r^2 = \langle\langle r_1(t)^2\rangle\rangle_t = A^2\,\frac{\phi^2}{\eta^2}\,\frac{e^{-\eta/\nu} + \eta/\nu - 1}{\eta/\nu}\,. \quad (6)$$

Using (3), the cross-correlation is also easily obtained as

$$\rho \equiv \frac{\langle r(t)F(t)\rangle}{\sigma_r\,A} = \sqrt{\frac{e^{-\eta/\nu} + \eta/\nu - 1}{\eta/\nu}}. \quad (7)$$

Expressions (6) and (7), valid for small values of A, are in excellent agreement with the numerical results. As shown in Fig. 2, we observe only quantitative deviations between theory and numerical results for large values of A: the peak in volatility and the vanishing of the $(r(t), F(t))$ cross-correlation are still present but with a shift towards smaller values of Q as A increases.

To show that the volatility amplification and de-correlation of response and driving force are robust with respect to the structure of the network, Fig. 3 shows σ_r/A and ρ as a function of Q for different networks. We consider a two-dimensional regular grid with Moore neighborhood and random small-world connections with varying concentration p_w. Changing p_w from 0 to 1 interpolates between the regular 2D lattice and the completely random network. For each p_w, the peak in volatility is still concomitant with the vanishing of ρ at some critical value $Q_c(p_w)$. This function $Q_c(p_w)$ is increasing in p_w, as larger global interconnection enhances the cooperative organization, and larger disorder is needed to destroy the ferromagnetic state.

Excess volatility in financial markets. In our model, the units are now the investors and the two states $s = \pm 1$ correspond to a positive or negative view on the future price move of the stock market. We assume for simplicity that traders invest in a single asset. A given investor forms a view on the asset according to expression (1), which combines the effect of external news, the trader's
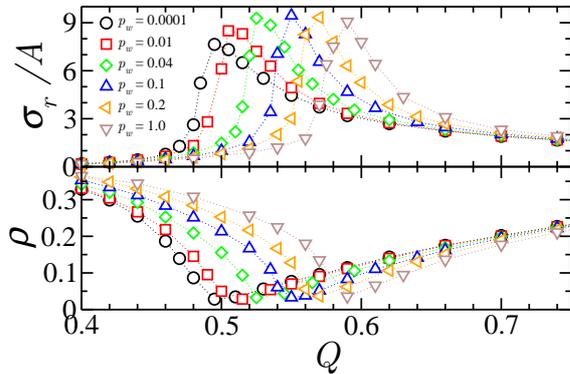
FIG. 3: (color online) As in Fig. 2, volatility σ_r/A and cross-correlation ρ as a function of the disorder Q for different small-world random connection concentrations p_w of a two-dimensional regular grid with Moore neighborhood. The other system parameters are $N = 10^4$, $k = 1.0$ and $A = 0.04$.

FIG. 4: (color online) Upper panel: Sample dynamics $r(t)$ (black bursty line) when $K(t)$ undergoes a confined random walk (green) in $[k - \Delta k; k + \Delta k]$ with $\Delta k = 0.5$ and step size $\sigma_k = \Delta k/\sqrt{5000}$. Red curve: featureless driving force $F(t)$. Lower-right panel: quickly vanishing (resp. long memory of) auto-correlation of $r(t)$ (thin lines) (resp. $|r(t)|$ (thick line)). Lower-left panel: DIV resonance in the presence of the time varying $K(t)$, with $\Delta k = 0.1$ (circles), 0.2 (squares), 0.5 (diamonds). The remaining parameters are $N = 10^4$, $A = 0.04$, $Q = 1$.

own judgment and the influence of his colleagues. At each micro-time step, a given trader i places an order to buy or sell a fixed asset quantity according to the sign $s_i(t)$ of his opinion. This order is fulfilled by a market maker, who acts as the counter party. Aggregating all the orders occurring at the micro-scale, the price dynamics at the macro-time level is assumed to follow $\log[p(t+1)] = \log[p(t)] + r(t+1)/\lambda$, where λ represents the liquidity depth of the market and is assumed constant. This equation expresses a linear market impact of the orders. The results below do not change qualitatively for more general non-linear impact functions [20].

To apply our model to the financial markets, we use k instead of Q as the control parameter. Rather than assuming a fixed coupling strength for investors, we propose that the impact of colleagues' opinions on a given trader may be both heterogeneous in the population and slowly varying with time. The later effect reflects varying perception of uncertainty, which is known to impact the propensity of humans to herd [21]. There are many varying sources of uncertainty that impact financial markets, including the economic and geopolitical climate and past stock market performance. In the spirit of Ref. [22], all these factors are embodied into the notion that $K(t)$ undergoes a slow random walk with i.i.d. increments $K(t + \delta t) - K(t) \sim N(0, \sigma_k)$, which is confined in the interval $[k - \Delta k; k + \Delta k]$. This later constraint ensures that social imitation remains bounded. We could have used an Ornstein-Uhlenbeck process or any other such confining dynamics, without changing the crucial results presented below.

By the mechanism of sweeping of the coupling strength $K(t)$ close to the critical point k_c (for fixed Q) [23], we expect and find transient burst of volatility amplifying a featureless random driving force $F(t)$. Fig. 4 shows a typical simulation, where the normalized return $r(t)$ exhibits transient bursts associated with excursion of $K(t)$

in the neighborhood of k_c. The lower-left panel of Fig 4 shows the robustness of the DIV phenomenon as a function of the average coupling k: even with a fluctuating $K(t)$, a large volatility peak appears for intermediate k. The lower-right panel shows very short-range correlations of $r(t)$ but very long-range correlations of the financial volatility $|r(t)|$ (another equivalent proxy for σ_r), very similar to empirical observations [24]. Such long persistence of the volatility can be traced back to the persistence of the confined random walk of $K(t)$: when the social interactions are strong (weak), they tend to remain strong (weak).

[1] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[2] A. S. Pikovsky and J. Kurths, Phys. Rev. Lett. **78**, 775 (1997).
[3] C. van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).
[4] P. Hänggi and F. Marchesoni, Rev. Mod. Phys. **81**, 387 (2009).
[5] D. Abbott, Fluct. Noise Lett. **9**, 129 (2010).
[6] J. J. Collins, C. C. Chow, and T. T. Imhoff, Nature **376**, 236 (1995).
[7] J. J. Brey, Phys. Lett. A **216**, 240 (1996).

[8] K. T. Leung and Z. Néda, Phys. Rev. E **59**, 2730 (1999).

[9] C. J. Tessone, C. R. Mirasso, R. Toral, and J. D. Gunton, Phys. Rev. Lett. **97**, 194101 (2006).

[10] J. A. Acebrón, S. Lozano, and A. Arenas, Phys. Rev. Lett. **99**, 128701 (2007).

[11] S. Ciliberto, A. Guarino, and R. Scorretti, Physica D **158**, 83 (2001).

[12] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[13] I. K. Ono *et al.*, Phys. Rev. Lett. **89**, 095703 (2002).

[14] R. B. Pandey and D. Stauffer, J. Stat. Phys. **61**, 235 (1990).

[15] S. Bewick, R. Yang, and M. Zhang, PLoS ONE **4**, e8112 (2009).

[16] D. Sornette, V. Yukalov, E. Yukalova, J.-Y. Henry, D. Schwab, and J. Cobb, J. Biol. Syst. **17**, 225 (2009).

[17] R. J. Shiller, Am. Econ. Rev. **71**, 421 (1981).

[18] E. F. Fama, J. Finance **25**, 383 (1970).

[19] R. Brealey, S. Myers, and A. Marcus, *Fundamentals of corporate finance* (McGraw-Hill Irwin, 2009), 6th ed.

[20] F. Lillo, J. D. Farmer, and R. N. Mantegna, Nature **421**, 129 (2003).

[21] S. Bikhchandani, D. Hirshleifer, and I. Welch, J. Pol. Econ. **100**, 992 (1992).

[22] D. Stauffer and D. Sornette, Physica A **271**, 496 (1999).

[23] D. Sornette, J. Phys. I France **4**, 209 (1994).

[24] Z. Ding, C. W. J. Granger, and R. F. Engle, J. Empirical Finance **1**, 83 (1993).