

A Dynamic Model of Network Formation with Strategic Interactions

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A Dynamic Model of Network Formation with Strategic Interactions[☆]

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1. Introduction

Social networks are important in several facets of our lives. For example, the decision of an agent of whether or not to buy a new product, attend a meeting, commit a crime, find a job is often influenced by the choices of his or her friends and acquaintances. The emerging empirical evidence on these issues motivates the theoretical study of network effects. For example, job offers can be obtained from direct and indirect, acquaintances through word-of-mouth communication. Also, risk-sharing devices and cooperation usually rely on family and friendship ties. Spread of diseases, such as AIDS infection, also strongly depends on the geometry of social contacts. If the web of connections is dense, we can expect higher infection rates. In terms of structure, real-life networks are characterized by low diameter (the so-called “small world” property), high clustering, and “scale-free” degree distributions.

To fathom these different aspects and to match the observed structure of real-life networks, one needs to analyze how and why networks form, the impact of network structure on agents’ outcomes, and the evolution of networks over time. The aim of the present paper is to propose a theoretical model that has all these features.

The literature on network formation is basically divided in two strands that are not communicating very much with each other. In the random network approach (mainly developed by mathematicians and physicists),¹ which is mainly dynamic, the reason why a link formed is pure chance. Indeed, this literature builds networks either through a purely stochastic process where links appear at random according to some distribution, or else through some algorithm for building links. In the other approach (developed by economists), which is mainly static, the reason for the formation of a link is strategic interactions. Individuals carefully decide with whom to interact and this decision entails some consent by both parts in a given relationship.² As Jackson [2007, 2008] pointed out, the random approach gives us a great deal of insight into *how* networks form (i.e. matches the characteristics of real-life networks) while the deterministic approach performs better on *why* networks form.

There is also another strand of the literature (called “games on net-

¹See Albert and Barabási [2002].

²Most of models of strategic network formation are static. Two prominent exceptions include Jackson and Watts [2002b] and Dutta et al. [2005]. Jackson and Watts [2002b] model network formation as an intertemporal process with *myopic* individuals breaking and forming links as the network evolves dynamically. Individuals are myopic in the sense that their decisions are guided completely by current payoffs, although the process of network formation takes place over real time. Dutta et al. [2005] relax this assumption and assume that agents behave in a farsighted manner by taking into account the intertemporal repercussions of their own decisions.

works”) that takes the network as given and studies how the network structure impacts on outcomes and individual decisions. A prominent paper of this literature is [Ballester et al. \[2006\]](#).³ They mainly show that if agents’ payoffs are linear-quadratic, then the unique interior Nash equilibrium of an n -player game in which agents are embedded in a network is such that each individual effort is proportional to her Bonacich centrality measure. The latter is a well-known centrality measure introduced by [Bonacich \[1987\]](#).⁴ In other words, it is mainly the centrality of an agent in a network that explains her outcome.⁵

To the best of our knowledge, there are very few papers that combine the literature on network formation and games on networks.⁶ The aim of this paper is to introduce strategic interactions in a non-random dynamic network formation game where agents also choose how much effort they put in their activities. By combining these approaches, we will also be able to match the characteristics of most real-life networks.

To be more precise, we develop a two-stage game where, in the first stage, as in [Ballester et al. \[2006\]](#), agents play their equilibrium contributions proportional to their Bonacich centrality while, in the second stage, a randomly chosen agent can update her linking strategy by creating a new link as a local best response to the current network. Furthermore, agents are embedded in a volatile environment which requires them to continually adapt to changing conditions. We assume that a link of a randomly selected agent decays, i.e. can be severed.

As a result, the formation of social networks can be regarded as a tension between the search for new linking opportunities and volatility that leads to the decay of existing links. Let us be more precise about each of the two

³ [Bramouille and Kranton \[2007\]](#) and [Galeotti et al. \[2009\]](#) are also important papers in this literature. The former focuses on local substitutabilities between agents connected in a network while the latter provides a model where agents do not have perfect information about the network. As in the present paper, [Ballester et al. \[2006\]](#) analyze a network game of local complementarities under perfect information

⁴Centrality is a fundamental measure of the importance of actors in social networks, dating back to early works such as [Bavelas \[1948\]](#). See [Wasserman and Faust \[1994\]](#) for an introduction and survey.

⁵In the empirical literature, it has been shown that centrality is important in explaining exchange networks [[Cook et al., 1983](#)], peer effects [[Calvó-Armengol et al., 2009](#); [Haynie, 2001](#)], creativity of workers [[Perry-Smith and Shalley, 2003](#)], workers’ performance [[Mehra et al., 2001](#)], power in organizations [[Brass, 1984](#)], the flow of information [[Borgatti, 2005](#); [Stephenson and Zelen, 1989](#)], the formation and performance of R&D collaborating firms and inter-organizational networks [[Boje and Whetten, 1981](#); [Powell et al., 1996](#); [Uzzi, 1997](#)] as well as the success of open-source projects [[Grewal et al., 2006](#)].

⁶Notable exceptions are [Bramouille et al. \[2004\]](#), [Cabral et al. \[2009\]](#), [Calvó-Armengol and Zenou \[2004\]](#), [Galeotti and Goyal \[2009\]](#), [Goyal and Vega-Redondo \[2005\]](#), [Jackson and Watts \[2002a\]](#). Contrary to our approach, all these models are static, and are unable to reproduce the main characteristics of real-world networks. Also, the network formation process is very different.

mechanisms: (i) link creation and (ii) link decay.

- (i) We assume that a randomly selected agent in the network creates a link with the agent with the highest centrality among the neighbors of her neighbors (the second order neighbors). This means that the value of each link is not an exogenous parameter but rather depends on the structure of the social network given by the centrality of an agent.⁷
- (ii) The volatility of the environment is an essential feature of our model. It may affect the value of a connection and, in turn, make it unprofitable. Moreover, volatility expresses the fact that there exist constraints on the number of links an agent can maintain. Similar to other authors (e.g. Ehrhardt et al. [2006b, 2008], Marsili et al. [2004], Vega-Redondo [2006]), we therefore assume that a link of a randomly selected agent decays. However, differently to these works, we do not assume that links decay at an exogenously given rate that is constant for all links connecting agents. Instead, we assume that agents view the links to the most central agents in their neighborhood as more valuable than the links to agents with low centrality. Under these conditions, agents use more valuable links more frequently. On the other hand, less frequently used links are exposed to stochastic link decay. As a result, less frequently used links decay before more frequently used links are disrupted.

We first show that, at each period of time, the network generated by this dynamic formation process is a nested split graph. These graphs, which are relatively well-known in the applied mathematics literature, have a very nice and simple structure that make them very tractable to work with. To the best of our knowledge, this is a first time that a complex dynamic network formation model can be characterized by such a simple structure in terms of networks it generates. By doing so, we are able to bridge the economics literature and the applied mathematics/physics literatures in a simple way. Because of their simple features, we then show that degree, closeness, and Bonacich centrality induce the same ordering of nodes in a nested split graph (this is also true for betweenness centrality if the ordering is not strict). This implies, in particular, that if we had a game where agents formed links according to other measures of centrality (such as degree, closeness, or betweenness) than the Bonacich centrality, then all our results would be unchanged. We then show that there exists a unique stationary

⁷ We further show that among all the possible links to second order neighbors the link to the one with the highest centrality increases the centrality (and thus the utility) of both agents (the initiator and the target of the link) the most. Thus agents do not only connect to agents with high centrality but they also strive to maximize their own centrality (and thus their own utility). In this broader sense we can view the link formation process as a competition for high centrality.

network, which is a nested split graph. We then demonstrate under which conditions these networks emerge and that there exists a sharp transition between hierarchical and flat network structures. Instead of relying on a mean-field approximation of the degree distribution and related measures as most dynamic network formation models do, because of the nature of nested split graphs, we are able to derive explicit solutions for all network statistics of the stationary network (by computing the adjacency matrix). We also observe that the network architecture adapts to changes in the volatility of the environment. We also find that, by altering the rate at which linking opportunities arrive and links decay, a sharp transition takes place in the network density. In line with previous works [Arenas et al., 2008; Guimerà et al., 2002; Visser, 2000], this transition entails a crossover from highly centralized networks when the linking opportunities are rare and the link decay is high to highly decentralized networks when many linking opportunities arrive and only few links are removed. From the efficiency perspective such sharp transition can also be observed in aggregate payoffs in stationary networks. It is important to observe that all our results are robust to any utility function as long as it increases in the number of links of each agent’s direct neighbors.

Finally, as in Jackson and Rogers [2007], we then proceed by showing that our model reproduces the main empirical observations of social networks. Indeed, we show that the stationary networks emerging in our link formation process are characterized by *short path length* with *high clustering* (so called “small worlds”, see Watts and Strogatz [1998]), *exponential degree distributions* with *power law tails* and *negative degree-clustering correlation*. These networks also show a clear core-periphery structure. Moreover, we show that, if agents have no “budget constraints” and can form any number of links then stationary networks are *dissortative*. However, if one takes into account capacity constraints in the number of links an agent can maintain, and allows for random global attachment between agents, we keep all the above mentioned network statistics while, at the same time, yielding *assortative* stationary networks.

The paper is organized as follows. In Section 2, we introduce the model and discuss the basic properties of the network formation process. In particular, Section 2.1 discusses the first stage of the game. In Section 2.2, we introduce the second stage of the game, where the network formation is explained. Next, Section 3 shows that stationary networks exist and can be computed analytically. After deriving the stationary networks, in Section 4, we analyze their properties in terms of topology and centralization. In Section 5, we study efficiency from the point of view of maximizing total efforts and aggregate payoff in the stationary network. We investigate the efficiency of different stationary networks as a function of the volatility of the environment. Section 6 discusses our results and their robustness, especially when we consider very general utility functions. Appendix A gives all the neces-

sary definitions and characterizations of general networks. In Appendix B, we focus on a class of networks (nested split graphs) that are important in our analysis and provide a general in terms of their topology properties and centralization measures. We extend our analysis in Appendix C by including capacity constraints in the number of links an agent can maintain and a global search mechanism for new linking partners. Finally, all proofs can be found in Appendix D.

2. The model

In this section, we develop a two-stage game. In the first stage, following Ballester et al. [2006], all agents simultaneously choose their effort level in a fixed network structure. It is a game with local complementarities where players have linear-quadratic payoff functions. In the second stage, a randomly chosen agent decides with whom she wants to form a link while a volatile environment forces the least frequently used link of a randomly selected agent to decay. This introduces two different time scales, one in which agents are choosing their efforts in a simultaneous move game and the second in which an agent forms a link as a best response to the current network. In which each agent chooses an effort level. We assume that the time in which agents are forming new links evolves much slower than the rate at which the stage game is repeated (see Vega-Redondo [2006], for a similar approach).

2.1. Nash Equilibrium and Bonacich Centrality

Consider a static network G in which the nodes represent a set of agents/players $N = \{1, 2, \dots, n\}$. Following Ballester et al. [2006], each agent $i \in N$ in the network G selects an effort level $x_i \geq 0$, $\mathbf{x} \in \mathbb{R}_+^n$, and receives a payoff $\pi_i(x_1, \dots, x_n)$ of the following form

$$\pi_i(x_1, \dots, x_n) = x_i - \frac{1}{2}x_i^2 + \lambda \sum_{j=1}^n a_{ij}x_ix_j. \quad (1)$$

This utility function is additively separable in the idiosyncratic effort component ($x_i - \frac{1}{2}x_i^2$) and the peer effect contribution ($\lambda \sum_{j=1}^n a_{ij}x_ix_j$). Payoffs display strategic complementarities in effort levels, i.e., $\partial^2 \pi_i(x_1, \dots, x_n) / \partial x_i \partial x_j = \lambda a_{ij} \geq 0$. In order to find the Nash equilibrium solution associated with the above payoff function, we define a network centrality measure introduced by Bonacich [1987]. Let \mathbf{A} be the symmetric $n \times n$ adjacency matrix of the network G and $\lambda_{\text{PF}}(G)$ its largest real eigenvalue. If \mathbf{I} denotes the $n \times n$ identity matrix and $\mathbf{u} = (1, \dots, 1)^T$ the n -dimensional vector of ones then we can define Bonacich centrality as follows:

Definition 1. The matrix $\mathbf{B}(G, \lambda) = (\mathbf{I} - \lambda\mathbf{A})^{-1}$ exists and is non-negative if and only if $\lambda < 1/\lambda_{PF}(G)$.⁸ Then

$$\mathbf{B}(G, \lambda) = \sum_{k=0}^{\infty} \lambda^k \mathbf{A}^k.$$

The Bonacich centrality vector is given by

$$\mathbf{b}(G, \lambda) = \mathbf{B}(G, \lambda) \cdot \mathbf{u}. \quad (2)$$

We can write the Bonacich centrality vector as

$$\mathbf{b}(G, \lambda) = \sum_{k=0}^{\infty} \lambda^k \mathbf{A}^k \cdot \mathbf{u} = (\mathbf{I} - \lambda\mathbf{A})^{-1} \cdot \mathbf{u}.$$

For the components $b_i(G, \lambda)$, $i = 1, \dots, n$, we get

$$b_i(G, \lambda) = \sum_{k=0}^{\infty} \lambda^k (\mathbf{A}^k \cdot \mathbf{u})_i = \sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^n (\mathbf{A}^k)_{ij}, \quad (3)$$

where $(\mathbf{A}^k)_{ij}$ is the ij -th entry of \mathbf{A}^k . Because $\sum_{j=1}^n (\mathbf{A}^k)_{ij}$ is the number of all walks of length k in G starting from i , $b_i(G, \lambda)$ is the number of all walks in G starting from i , where the walks of length k are weighted by their geometrically decaying factor λ^k .

Now we can turn to the equilibrium analysis of the game.

Theorem 1 (Ballester et al. [2006]). Let $\mathbf{b}(G, \lambda)$ be the Bonacich network centrality of parameter λ . For $\lambda < \lambda_{PF}(G)$, the unique interior Nash equilibrium solution of the simultaneous n -player move game with payoffs given by (1) and strategy space \mathbb{R}_+^n is given by

$$x_i^* = b_i(G, \lambda), \quad (4)$$

for all $i = 1, \dots, n$.

Moreover, the payoff of agent i in the equilibrium is given by

$$\pi_i(\mathbf{x}^*, G) = \frac{1}{2}(x_i^*)^2 = \frac{1}{2}b_i^2(G, \lambda). \quad (5)$$

The parameter λ measures the effect on agent i of agent j 's contribution, if they are connected. If we assume that we have strong network externalities so that λ approaches its highest possible value $1/\lambda_{PF}(G)$ then the Bonacich

⁸The proof can be found e.g. in [Debreu and Herstein \[1953\]](#).

centrality becomes proportional to the standard eigenvector measure of centrality [Wasserman and Faust, 1994]. The latter result has been shown by Bonacich [1987] and Bonacich and Lloyd [2001].

Furthermore, Ballester et al. [2006] have shown that the equilibrium outcome and the payoff for each player increases with the number of links in G (because the number of network walks increases in this way).⁹ This implies that, if an agent is given the opportunity to change her links, she will add as many links as possible. On the other hand, if she is only allowed to form one link at a time, she will form the link to the agent that increases her payoff the most. In both cases, eventually, the network will then become complete, i.e. each agent is connected to every other agent. However, to avoid this latter unrealistic situation, we assume that the agents are living in a volatile environment that causes links to decay such that the complete network can never be reached. Instead the architecture of the network adapts to the volatile environment. We will treat these issues more formally in the next section.

2.2. The Network Formation Game

We now introduce a network formation process that incorporates the idea that agents with high Bonacich centrality (their equilibrium effort levels) are more likely to connect to each other and that the presence of common neighbors enhances the likelihood of agents to form a new link between them.

Let time be measured at countable dates $t = 0, 1, 2, \dots$ and consider the network formation process $(G(t))_{t=0}^{\infty}$ with $G(t) = (N, L(t))$ comprising the set of agents $N = \{1, \dots, n\}$ together with the set of links $L(t)$ at time t .¹⁰ The timing is as follows: At $t = 0$, we start with the empty network $G(0) = \bar{K}_n$. Then every agent $i \in N$ optimally chooses her effort $x_i \in \mathbb{R}_+$, which is $x_i^* = 1$, since $b_i(\bar{K}_n, \lambda) = 1$ for all $i = 1, \dots, n$.¹¹ Then, an agent i is chosen at random and with probability $p_i \in (0, 1)$ forms a link with agent j that gives her the highest utility (or equivalently her highest Bonacich centrality). We obtain the network $G(1)$. Then, again, a player i is chosen at random and with probability p_i decides with whom she wants to form a link. For that, she has to calculate all the possible network configurations and chooses the one that gives her the highest utility. An so forth.

Let us now explain the game in more detail and, in particular, the formation of links between agents. Let $\mathcal{N}_i = \{k \in N : ik \in L(t)\}$ be the set of neighbors of agent $i \in N$ and $\mathcal{N}_i^{(2)} = \bigcup_{j \in \mathcal{N}_i} \mathcal{N}_j \setminus (\mathcal{N}_i \cup \{i\})$ denote the second-order neighbors of agent i in the current network $G(t)$. We assume that agents form links only with the neighbors of their neighbors. Quite

⁹See Theorem 2 in Ballester et al. [2006].

¹⁰We give a formal characterization of this stochastic process in Section 3.

¹¹The adjacency matrix \mathbf{A} of the empty network \bar{K}_n contains only zero entries and therefore $\mathbf{b}(\bar{K}_n, \lambda) = (\mathbf{I} - \lambda\mathbf{A})^{-1}\mathbf{u} = \mathbf{u}$.

naturally, if an agent has no links, then she will search among all agents for the best links. We make this assumption because agents know mainly their friends and the friends of their friends. In the friendship example, individuals connect to friends of friends because they trust their own friends who can recommend them to their acquaintances. Also, each individual is likely to meet a friend of friend and thus decides to create a link or not. More generally, it seems reasonable that the formation of links is limited to agents that someone is aware of. This should be even more true in large networks where players' information may be limited to their immediate "neighborhood".¹²

The key question is how individuals choose among their second-order neighbors (i.e. friends of friends). Let us explain the way someone is selected to form a link. At every t , an agent i , selected uniformly at random from the set N , enjoys an updating opportunity of her current links at a rate p_i . If an agent receives such an opportunity, then she initiates a link to agent j which increases her equilibrium payoff the most in her second-order neighborhood $\mathcal{N}_i^{(2)}$. Agent j is said to be the *local best response* of agent i given the network $G(t)$. Agent j accepts the link if i has also the highest centrality in her second-order neighborhood $\mathcal{N}_j^{(2)}$. That is, agent i is also a local best response of agent j . The underlying assumption for this is that individuals carefully decide with whom to interact and this decision entails some consent by both parts in a given relationship. Note, that the connectivity relation is symmetric such that j is a second-order neighbor of i if i is a second order neighbor of j , i.e. $i \in \mathcal{N}_j^{(2)}$ if and only if $j \in \mathcal{N}_i^{(2)}$ for all $i, j \in N$. Moreover, as we will see below, agent i is always a local best response of agent j if agent j is a local best response of agent i .

Observe that when agents decide to create a link, they do it in a *myopic* way, that is they only look at the second-order neighbor that gives them the *current* highest utility. There is literature on farsighted networks where agents calculate their lifetime-expected utility when they want to create a link (see, e.g. [Konishi and Ray \[2003\]](#)). We adopt here a myopic approach because of its tractability and because our model also incorporates effort decision.¹³

Let us give a formal definition of the local best responses of an agent given the prevailing network $G(t)$.¹⁴

¹²In Appendix C, we allow agents to create links with agents further away in the network, i.e. at length greater than two.

¹³[Jackson and Watts \[2002b\]](#) argue that this form of myopic behavior makes sense if players discount heavily the future.

¹⁴In order to guarantee an interior solution of the Nash equilibrium efforts corresponding to the payoff functions in equation (1), we assume that the parameter λ is smaller than the inverse of the largest real eigenvalue of $G(t)$ for any t . Testing the impact of the Bonacich centrality measure on educational outcomes in the United States, [Calvó-Armengol et al. \[2009\]](#) found that only 18 out of 199 networks (i.e. 9 percent) do not satisfy this eigenvalue

Definition 2. Consider the current network $G(t) = (N, L(t))$ with agents $N = \{1, \dots, n\}$ and links $L(t)$. Let $G(t) + ij$ be the graph obtained from $G(t)$ by the addition of the edge $ij \notin L(t)$ between agents $i, j \in N$. Further, let $\pi^*(G(t)) = (\pi_1^*(G(t)), \dots, \pi_n^*(G(t)))$ denote the profile of Nash equilibrium payoffs of the agents in $G(t)$ following from the payoff function (1) with parameter $\lambda < 1/\lambda_{PF}(G(t))$. Then agent j is a local best response of agent i if $\pi_i^*(G(t) + ij) \geq \pi_i^*(G(t) + ik)$ for all $j, k \in \mathcal{N}_i^{(2)}$. Agent j may not be unique. The set of agent i 's local best responses is denoted by $BR_i(G(t))$. If agent i does not have any second-order neighbors, $\mathcal{N}_i^{(2)} = \emptyset$, then agent j is a local best response of agent i if $\pi_i^*(G(t) + ij) \geq \pi_i^*(G(t) + ik)$ for all $j, k \in N \setminus (\mathcal{N}_i \cup \{i\})$.

Note that the best response strategies for the network games introduced in Bala and Goyal [2000]; Haller et al. [2007]; Haller and Sarangi [2005] allow an agent to remove or create an arbitrary number of links while we restrict the link formation (strategy space) of an agent to one additional link only. We omit the removal of links since agents payoffs are monotonic increasing in the number of links in the network. Since the removal of a link would always decrease an agent's payoffs, link removal is strictly dominated by link creation.

We assume that during the time interval from t to $t + 1$ an agent i is selected and either has the possibility to create a link (with probability $p_i \in (0, 1)$) or to sever a link (with probability $q_i \in (0, 1)$). Note that taking into account the possibility of an agent remaining quiescent only modifies the time-scale of the process discussed, thus yielding identical results to the model proposed. This implies that, without any loss of generality, it is possible to assume $p_i + q_i = 1$. For simplicity, we also assume that these probabilities are the same across agents. Accordingly, we will use one parameter α and $1 - \alpha$ to denote the probabilities at which links are formed and removed respectively, that is, $p_i = \alpha$ and $q_i = 1 - \alpha$, with $\alpha \in (0, 1)$, for all $i \in N$.

Definition 3. We define the network formation process $(G(t))_{t=0}^\infty$, $G(t) = (N, L(t))$, as a sequence of networks $G(0), G(1), G(2), \dots$ in which at every step $t = 0, 1, 2, \dots$, an agent $i \in N$ is uniformly selected at random. Then one of the following two events occurs:

- (i) With probability $\alpha \in (0, 1)$ agent i initiates a link to a local best response agent $j \in BR_i(G(t))$. Then the link ij is created if $i \in BR_j(G(t))$ is a local best response of j , given the current network $G(t)$. If $BR_i(G(t)) = \emptyset$ or $BR_j(G(t)) = \emptyset$ nothing happens. If $BR_i(G(t))$ is not unique, then i selects randomly one agent in $BR_i(G(t))$.

condition.

- (ii) With probability $1 - \alpha$ the link $ij \in L(t)$ is removed such that $\pi_i^*(G(t) - ij) \geq \pi_i^*(G(t) - ik)$ for all $j, k \in \mathcal{N}_i$. If agent i does not have any link then nothing happens.

In words, with probability α , the selected agent will create a link with her second-order neighbor who increases the most her utility, while with probability $1 - \alpha$, the selected agent will delete a link with her direct neighbor who reduces the least her utility. This link is for the selected agent the least important and thus the least frequently used. Note that the newly established link also affects the overall network structure and therewith the centralities and payoffs of all other agents (in the same connected component). The formation of links thus can introduce large, unintended and uncompensated externalities.

2.3. Network Formation and Nested Split Graphs

An essential property of the link formation process $(G(t))_{t=0}^{\infty}$ introduced in Definition 3 is that it produces networks in a well defined class of graphs denoted by “nested split graphs” [Aouchiche et al., 2006].¹⁵ We will give a formal definition of these graphs and discuss an example in this section. Nested split graphs include many common networks such as the star or the complete network. Moreover, as their name already indicates, they have a *nested neighborhood structure*. This means that the set of neighbors of each agent is contained in the set of neighbors of each higher degree agent. Nested split graphs have particular topological properties and an associated adjacency matrix with a well defined structure.

In order to characterize nested split graphs, it will be necessary to consider the degree partition of a graph, which is defined as follows:

Definition 4 (Mahadev and Peled [1995]). Let $G = (N, L)$ be a graph whose distinct positive degrees are $d_{(1)} < d_{(2)} < \dots < d_{(k)}$, and let $d_0 = 0$ (even if no agent with degree 0 exists in G). Further, define $D_i = \{v \in N : d_v = d_{(i)}\}$ for $i = 0, \dots, k$. Then the vector $\mathbf{D} = (D_0, D_1, \dots, D_k)$ is called the *degree partition* of G .

With the definition of a degree partition, we can now give a more formal definition of a nested split graph.^{16,17}

¹⁵Nested split graphs are also called “threshold networks” [Hagberg et al., 2006; Mahadev and Peled, 1995].

¹⁶ Let x be a real valued number $x \in \mathbb{R}$. Then, $\lceil x \rceil$ denotes the smallest integer larger or equal than x (the ceiling of x). Similarly, $\lfloor x \rfloor$ denotes the largest integer smaller or equal than x (the floor of x).

¹⁷In general, split graphs are graphs whose nodes can be partitioned in a set of nodes which are all connected among each other and sets of nodes which are disconnected. A nested split graph is a special case of a split graph.

Definition 5 (Mahadev and Peled [1995]). Consider a nested split graph $G = (N, L)$ and let $\mathbf{D} = (D_0, D_1, \dots, D_k)$ be its degree partition. Then the nodes N can be partitioned in independent sets D_i , $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$ and dominating sets D_i , $i = \lfloor \frac{k}{2} \rfloor + 1, \dots, k$. Moreover, the neighborhoods of the nodes are nested. In particular, for each node $v \in D_i$, $i = 1, \dots, k$,

$$N_v = \begin{cases} \bigcup_{j=1}^i D_{k+1-j} & \text{if } i = 1, \dots, \lfloor \frac{k}{2} \rfloor, \\ \bigcup_{j=1}^i D_{k+1-j} \setminus \{v\} & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \dots, k. \end{cases} \quad (6)$$

Figure 1 (left) illustrates the degree partition $\mathbf{D} = (D_0, D_1, \dots, D_6)$ and the nested neighborhood structure of a nested split graph. A line between D_i and D_j indicates that every node in D_i is linked to every node in D_j for any $i, j = 1, \dots, 6$. The nodes in the dominating sets included in the solid frame induce a clique while the nodes in the independent sets that are included in the dashed frame induce an empty subgraph.

A nested split graph has an associated adjacency matrix which is called *stepwise matrix* and it is defined as follows:

Definition 6 (Brualdi and Hoffman [1985]). A stepwise matrix \mathbf{A} is a matrix with elements a_{ij} satisfying the condition: if $i < j$ and $a_{ij} = 1$ then $a_{hk} = 1$ whenever $h < k \leq j$ and $h \leq i$.

Figure 1 (right) shows the stepwise adjacency matrix \mathbf{A} corresponding to the nested split graph shown on the left hand side. If we let the nodes be indexed by the order of the rows in the adjacency matrix \mathbf{A} then it is easily seen that for example $D_6 = \{1, 2 \in N : d_1 = d_2 = d_{(6)} = 9\}$ and $D_1 = \{9, 10 \in N : d_9 = d_{10} = d_{(1)} = 2\}$.

If a nested split graph is connected we call it a connected nested split graph. The representation and the adjacency matrix depicted in Figure 1 actually shows a connected nested split graph. From the stepwise property of the adjacency matrix it follows that a connected nested split graph contains at least one spanning star, that is, there is at least one agent that is connected to all other agents. In Appendix B, we also derive the clustering coefficient, the neighbor connectivity and the characteristic path length of a nested split graph. In particular, we show that connected nested split graphs have small characteristic path length, which is at most two. We also analyze different measures of centrality (see Wasserman and Faust [1994]) in a nested split graph. One important result is that degree, closeness, and Bonacich centrality induce the same ordering of nodes in a nested split graph. If the ordering is not strict, then this holds also for betweenness centrality (see Section B.2.5 in the Appendix).

In the next proposition, we identify the relationship between the Bonacich centrality of an agent and her degree in a nested split graph.

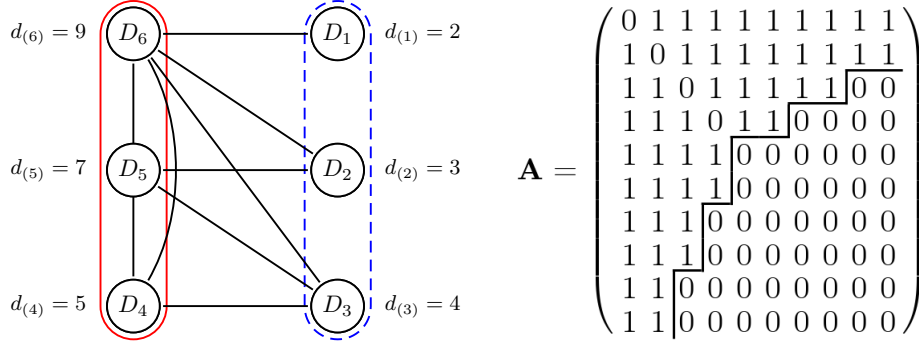


Figure 1: Representation of a connected nested split graph (left) and the associated adjacency matrix (right) with $n = 10$ agents and $k = 6$ distinct positive degrees. A line between D_i and D_j indicates that every node in D_i is linked to every node in D_j . The solid frame indicates the dominating sets and the nodes in the independent sets are included in the dashed frame. Next to the set D_i the degree of the nodes in the set is indicated. The neighborhoods are nested such that the degrees are given by $d_{(i+1)} = d_{(i)} + |D_{k-i+1}|$ for $i \neq \lfloor \frac{k}{2} \rfloor$ and $d_{(i+1)} = d_{(i)} + |D_{k-i+1}| - 1$ for $i = \lfloor \frac{k}{2} \rfloor$. In the corresponding adjacency matrix \mathbf{A} to the right the zero-entries are separated from the one-entries by a stepfunction.

Proposition 1. Consider a pair of agents $i, j \in N$ of a nested split graph $G = (N, L)$.

- (i) If and only if agent i has a higher degree than agent j then i has a higher Bonacich centrality than j , i.e.

$$d_i > d_j \Leftrightarrow b_i(G, \lambda) > b_j(G, \lambda).$$

- (ii) Assume that neither the links ik nor ij are in G , $ij \notin L$ and $ik \notin L$. Further assume that agent k has a higher degree than agent j , $d_k > d_j$. Then adding the link ik to G increases the Bonacich centrality of agent i more than adding the link ij to G , i.e.

$$d_k > d_j \Leftrightarrow b_i(G + ik, \lambda) > b_i(G + ij, \lambda).$$

From part (ii) of Proposition 1 we find that when agent i has to decide to create a link either to agents k or j , with $d_k > d_j$, in the link formation process $(G(t))_{t=0}^{\infty}$ then i will always connect to agent k because this link gives i a higher Bonacich centrality than the other link to agent j . We can make use of this property in order to show that the networks emerging from the link formation process defined in the previous section actually are nested split graphs. This result is stated in the next proposition.

Proposition 2. *Consider the network formation process $(G(t))_{t=0}^{\infty}$ introduced in Definition 3. Then, at any time $t \geq 0$, a network $G(t)$ is a nested split graph.*

This result is due to the fact that agents, when they have the possibility of creating a new link, always connect to the agent who has the highest Bonacich centrality (and by Proposition 1 the highest degree) among her second-order neighbors. This creates a nested neighborhood structure which can always be represented by a stepwise adjacency matrix after a possible relabeling of the agents.¹⁸ The same applies for link removal.

From the fact that $G(t)$ is a nested split graph with an associated stepwise adjacency matrix it further follows that at any time t in the network evolution, $G(t)$ consists of a single connected component and possibly isolated nodes.

Corollary 1. *Consider the network formation process $(G(t))_{t=0}^{\infty}$ introduced in Definition 3. Then, at any time $t \geq 0$, a network $G(t)$ consists of a connected component and possibly isolated nodes.*

Nested split graphs are not only prominent in the literature on spectral graph theory [Cvetkovic et al., 1997] but they have also appeared in the recent literature on economic networks. Nested split graphs are so called “inter-linked stars” found in Goyal and Joshi [2003].¹⁹ Subsequently, Goyal et al. [2006] identified inter-linked stars in the network of scientific collaborations among economists. It is important to note that nested split graphs are characterized by a distinctive core-periphery structure. Core-periphery structures have been found in several empirical studies of interfirm collaborations networks [Baker et al., 2008]. The wider applicability of nested split graphs suggests that a network formation process that generates these graphs as it is defined in Definition 3 are of general relevance for understanding economic and social networks.

3. Stationary Networks: Characterization

In this section we analyze in more detail the network formation process $(G(t))_{t=0}^{\infty}$ defined in the previous section, where $G(t)$ is the random variable

¹⁸Two graphs $G = (N, L)$ and $G' = (N', L')$ are the same unlabeled graph when they are isomorphic, i.e., when there exists a permutation $\pi: N \rightarrow N'$ such that $ij \in L$ if and only if $\pi(i)\pi(j) \in L'$. Further, we will show in Proposition 3 that $(G(t))_{t=0}^{\infty}$ induces a finite state Markov chain with state space Ω consisting of all unlabeled nested split graphs. Consequently, two states $x, y \in \Omega$ of the Markov chain $(G(t))_{t=0}^{\infty}$ are identical, $x = y$, if they correspond to the same unlabeled graph.

¹⁹Nested split graphs are inter-linked stars but an inter-linked star is not necessarily a nested split graph. Nested split graphs have a nested neighborhood structure for all degrees while in an inter-linked star this holds only for the nodes with the lowest and highest degrees.

realized at time $t \geq 0$. Let \mathcal{F} define the σ -algebra $\sigma(G(t) : t \in \mathbb{N}_0)$ generated by $G(0), G(1), \dots$ and let Ω denote the state space of $(G(t))_{t=0}^\infty$. Then the probability space is given by the triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is the probability measure satisfying $\sum_{G \in \Omega} \mathbb{P}(G) = 1$. In the following, we show that the network formation process $(G(t))_{t=0}^\infty$ induces an ergodic Markov chain and we analyze the asymptotic states of this process as the number n of agents becomes large.

Proposition 3. *The network formation process $(G(t))_{t=0}^\infty$ introduced in Definition 3 induces an ergodic Markov chain on the state space Ω with a unique stationary distribution μ . In particular, the state space Ω is finite and consists of all possible unlabeled nested split graphs on n nodes, where the number of possible states is given by $|\Omega| = 2^{n-1}$.*

The symmetry of the network formation process $(G(t))_{t=0}^\infty$ with respect to the link formation probability α and the link removal probability $1 - \alpha$ allows us to state the following proposition.

Proposition 4. *Consider the network formation process $(G(t))_{t=0}^\infty$ with link creation probability α and the network formation process $(G'(t))_{t=0}^\infty$ with link creation probability $1 - \alpha$. Let μ be the stationary distribution of $(G(t))_{t=0}^\infty$ and μ' the stationary distribution of $(G'(t))_{t=0}^\infty$. Then for each network G in the stationary distribution μ with probability μ_G the complement of G , \bar{G} , has the same probability μ_G in μ' , i.e. $\mu'_{\bar{G}} = \mu_G$.*

Proposition 4 allows us to derive the stationary distribution μ for any value of $1/2 < \alpha < 1$ if we know the corresponding distribution for $1 - \alpha$. This follows from the fact that the complement \bar{G} of a nested split graph G is a nested split graph as well [Mahadev and Peled, 1995]. In particular, the networks \bar{G} are nested split graphs in which the number of nodes in the dominating sets corresponds to the number of nodes in the independent sets in G and, conversely, the number of nodes in the independent sets in \bar{G} corresponds to the number of nodes in the dominating sets in G .

With this symmetry in mind we restrict our analysis in the following to the case of $0 < \alpha \leq 1/2$. Let $\{N(t)\}_{t=0}^\infty$ be the degree distribution with the d -th element $N_d(t)$, giving the number of nodes with degree d in $G(t)$, in the t -th sequence $N(t) = \{N_d(t)\}_{d=0}^{n-1}$. Further, let $n_d(t) = N_d(t)/n$ denote the proportion of nodes with degree d and let $n_d = \lim_{t \rightarrow \infty} \mathbb{E}(n_d(t))$ be its asymptotic expected value (as given by μ). In the following proposition we determine the asymptotic degree distribution of the nodes in the independent sets for n large enough.

Proposition 5. *Let $0 < \alpha \leq 1/2$. Then the asymptotic expected proportion n_d of nodes in the independent sets with degrees, $d = 0, 1, \dots, d^*$, for large n is given by*

$$n_d = \frac{1 - 2\alpha}{1 - \alpha} \left(\frac{\alpha}{1 - \alpha} \right)^d, \quad (7)$$

where²⁰

$$d^*(n, \alpha) = \frac{\ln \left(\frac{(1-2\alpha)n}{2(1-\alpha)} \right)}{\ln \left(\frac{1-\alpha}{\alpha} \right)}. \quad (8)$$

The structure of nested split graphs implies that if there exist nodes for all degrees between 0 and d^* (in the independent sets), then the dominating sets with degrees larger than d^* contain only a single node. Further, using Proposition 4, we know that for $\alpha > 1/2$ the expected number of nodes in the dominating sets is given by the expected number of nodes in the independent sets in Equation (7) for $1 - \alpha$, while each of the independent sets contains a single node. This determines the asymptotic degree distribution for the independent or dominating sets, respectively, for all values of α in the limit of large n .

Moreover, we can show that the empirical degree distribution converges in probability to the expected distribution in the limit of large network sizes n .

Proposition 6. *For any $\epsilon > 0$ we have that*

$$P(|n_d(t) - E(n_d(t))| \geq \epsilon) \leq 2e^{-\frac{\epsilon^2 n^2}{8t}}. \quad (9)$$

Furthermore, from Equation (8) we can directly derive the following corollary.

Corollary 2. *There exists a phase transition in the asymptotic average number of independent sets, $d^*(n, \alpha)$, as n becomes large such that*

$$\lim_{n \rightarrow \infty} \frac{d^*(n, \alpha)}{n} = \begin{cases} 0, & \text{if } \alpha < \frac{1}{2}, \\ \frac{1}{2}, & \text{if } \alpha = \frac{1}{2}, \\ 1, & \text{if } \alpha > \frac{1}{2}. \end{cases} \quad (10)$$

Corollary 2 implies that as n grows without bound the networks in the stationary distribution μ are either sparse or dense, depending on the value

²⁰Note that $d^*(n, \alpha)$ from Equation (8) might in general not be an integer. In this case we take the closest integer value to Equation (8), that is, we take $[d^*(n, \alpha)] = \lfloor d^*(n, \alpha) + \frac{1}{2} \rfloor$. The error we make in this approximation is negligible for large n .

of the link creation probability α . Moreover, from the functional form of $d(n, \alpha)$ in Equation (8) we find that there exists a sharp transition from sparse to dense networks as α crosses $1/2$ and the transition becomes sharper the larger is n .

Observe that, because a nested split graph is uniquely defined by its degree distribution²¹, Proposition 5 delivers us a complete description of a typical network generated by our model in the limit of large t and n . We call this network the “stationary network”. We can compute the degree distribution and the corresponding adjacency matrix of the stationary network for different values of α .²² The latter is shown in Figure 2. From the structure of these matrices we observe the transition from sparse networks containing a hub and many agents with small degree to a quite homogeneous network with many agents having similar high degrees. Moreover, this transition is sharp around $\alpha = 1/2$. In Figure 3, we show particular networks arising from the network formation process for the same values of α . Again, we can identify the sharp transition from hub-like networks (inter-linked stars) to homogeneous, almost complete networks.

Figure 4 displays the number of links m and the number of distinct degrees k as a function of α . We see that there exists a sharp transition from sparse to dense networks around $\alpha = 1/2$ while k reaches a maximum at $\alpha = 1/2$. This follows from the fact that $k = 2d^*$ with d^* given in Equation (8) is monotonic increasing in α for $\alpha < 1/2$ and monotonic decreasing in α for $\alpha > 1/2$.

4. Stationary Networks: Statistics

There exists a growing number of empirical studies trying to identify the key characteristics of social and economic networks. However, only few theoretical models (a notable exception is Jackson and Rogers [2007]) have tried to reproduce these findings to the full extent. We pursue the same approach. We show that our network formation model leads to properties which are shared with empirical networks. These properties can be summarized as follows:²³

- (i) The average shortest path length between pairs of agents is small [Albert and Barabási, 2002].
- (ii) Empirical networks exhibit high clustering [Watts and Strogatz, 1998].
This means that the neighbors of an agent are likely to be connected.

²¹The degree distribution uniquely determines the corresponding nested split graph up to a permutation of the indices of nodes.

²²Non-integer values for the partition sizes can be approximated with the closest integer while preserving the nested structure of the degree partitions.

²³This list of empirical regularities is far from being extensive and summarizes only the most pervasive patterns found in the literature.

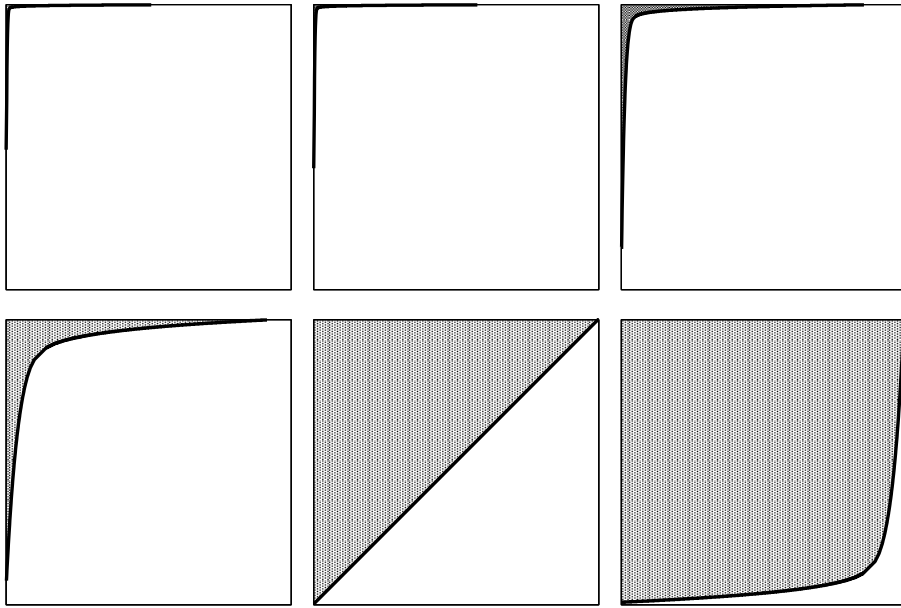


Figure 2: Representation of the adjacency matrices of stationary networks with $n = 1000$ agents for different values of parameter α : $\alpha = 0.2$ (top-left plot), $\alpha = 0.4$ (top-center plot), $\alpha = 0.48$ (top-right plot), $\alpha = 0.495$ (bottom-left plot), $\alpha = 0.5$ (bottom-center plot), and $\alpha = 0.52$ (bottom-right plot). The matrix top-left for $\alpha = 0.4$ is corresponding to an inter-linked star while the matrix bottom-right for $\alpha = 0.52$ corresponds to an almost complete network. Thus, there exists a sharp transition from sparse to densely connected stationary networks around $\alpha = 0.5$. Networks of smaller size for the same values of α can be seen in Figure 3.

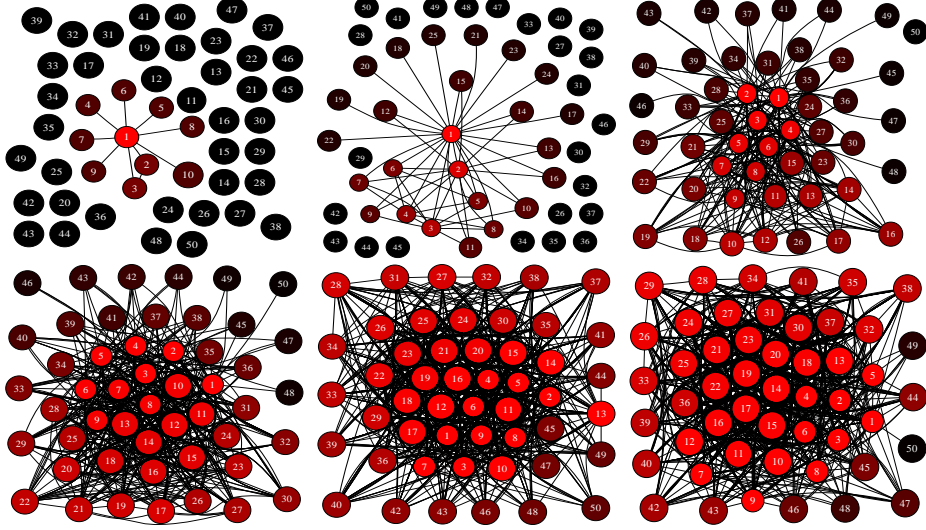


Figure 3: Sample networks with $n = 50$ agents for different values of parameter α : $\alpha = 0.2$ (top-left plot), $\alpha = 0.4$ (top-center plot), $\alpha = 0.48$ (top-right plot), $\alpha = 0.495$ (bottom-left plot), $\alpha = 0.5$ (bottom-center plot), and $\alpha = 0.52$ (bottom-right plot). Nodes with brighter shapes correspond to agents with a higher eigenvector centrality. The networks for small values of α are characterized by the presence of a hub and a growing cluster attached to the hub. With increasing values of α the density of the network increases until the network becomes almost complete. The network plots have been generated using a Fruchterman-Reingold algorithm [Fruchterman and Reingold, 1991].

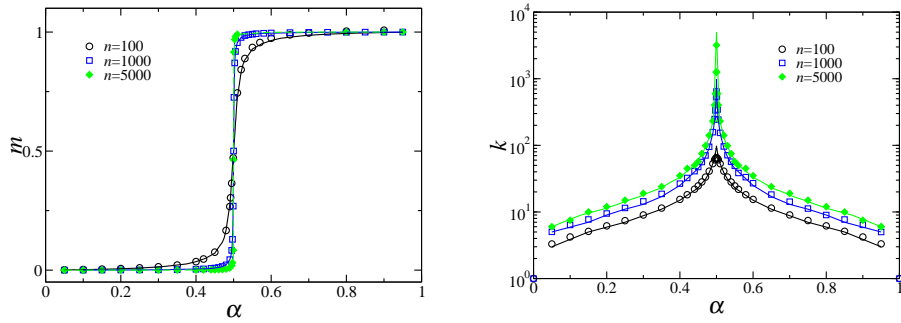


Figure 4: In the left panel we show the number of links m of the stationary network. The number of distinct degrees $k = 2d^*$, with d^* from Equation (8), found in the stationary network for different values of α are shown in the right panel. The figures display both, the results obtained by recourse of numerical simulations (symbols) and respecting theoretical predictions (lines) of the model.

- (iii) The distribution of degrees is highly skewed. While some authors [Barabasi and Albert \[1999\]](#) find power law degree distributions, others find deviations from power-laws in empirical networks, e.g. in [Newman \[2004\]](#).
- (iv) Several authors have found that there exists an inverse relationship between the clustering coefficient of an agent and her degree [[Goyal et al., 2006](#); [Pastor-Satorras et al., 2001](#)]. The neighbors of a high degree agent are less likely to be connected among each other than the neighbors of an agent with low degree. This means that empirical networks are characterized by a negative clustering-degree correlation.
- (v) Networks in economic and social contexts exhibit degree-degree correlations. [Newman \[2002, 2003\]](#) has shown that many social networks tend to be positively correlated. In this case the network is said to be assortative. On the other hand, technological networks such as the internet [[Pastor-Satorras et al., 2001](#)] display negative correlations. In this case the network is said to be disassortative. Others, however, find also negative correlations in social networks such as in the Ham radio network of interactions between amateur radio operators [[Killworth and Bernard, 1976](#)] or the affiliation network in a Karate club [[Zachary, 1977](#)]. Networks in economic contexts may have features of both technological and social relationships [[Jackson, 2008](#)] and so there exist examples with positive degree correlations such as in the network between venture capitalists [[Mas et al., 2007](#)] as well as negative degree correlations as it can be found in the world trade web [[Serrano and Boguñá, 2003](#)], online social communities [[Hu and Wang, 2009](#)] and in networks of banks [[De Masi and Gallegati, 2007](#); [May et al., 2008](#)].

In the following sections, we analyze some of the topological properties of the stationary networks in our model. With the asymptotic expected degree distribution derived in Proposition 5, we can calculate the expected clustering coefficient, the clustering-degree correlation, the neighbor connectivity, the assortativity, and the characteristic path length by using the expressions derived for these quantities in Appendix B, where we show that these statistics are all functions of the degree distribution.²⁴ These network measures are interesting because they can be compared to key empirical findings of social and economic networks. In fact, we show that the stationary networks exhibit all the well-known stylized facts of real-world networks. Moreover,

²⁴By virtue of Proposition 6, we know that the probability limit of the degree distribution is its expected value. Therefore we can compute the probability limit of these statistics in good approximation, by evaluating them at the expected degree distribution. In particular, we compute the integer valued sizes of the degree partition from the real-valued asymptotic degree distribution by taking closest integers.

we show in Appendix C that, by introducing capacity constraints in the number of links an agent can maintain and the possibility that links can be formed outside the neighbors' neighbors, we are able to produce both, assortative as well as disassortative networks.

Note that since the stationary distribution μ is unique, we can recover the expected value of any statistic by averaging over a large enough sample of empirical networks generated by numerical simulations. We then superimpose the analytical predictions of the statistic derived from Proposition 5 with the sample averages in order to compare the validity of our theoretical results, also for small network sizes n . As we will show, there is a good agreement of the theory with the empirical results for all network sizes.

4.1. Degree Distribution

From Proposition 5, we find that the degree distribution follows an exponential decay with a power-law tail.²⁵ The power-law tail has an exponent of minus one. Degree distributions with exponential and power-law parts have been found in empirical networks, e.g. in scientific collaboration networks Newman [2004]. For $\alpha = 1/2$ the degree distribution is uniform while for larger values of α most of the agents have a degree close to the maximum degree.

4.2. Clustering

The clustering coefficient is shown in Figure 6 (left). We find that for practically all values of α , the clustering in the stationary networks is high. This finding is in agreement with the vast literature on social networks that have reported high clustering being a distinctive feature of social networks. Moreover, Goyal et al. [2006] have shown that there exists a negative correlation between the clustering coefficient of an agent and her degree. We find

²⁵For $0 < \alpha \leq 1/2$ and n large enough the asymptotic expected degree distribution for the degrees d smaller or equal than d^* is given by an exponential function

$$n(d) = \frac{1 - 2\alpha}{1 - \alpha} e^{-\ln(\frac{1-\alpha}{\alpha})d}.$$

On the other hand, if we assume (i) that the degree of a dominating node is symmetrically distributed around its expected value, (ii) we compute the integral over the probability density function by a rectangle approximation and (iii) further assume that the degree distribution obtained in this way has the same functional form for all degrees d larger than d^* then one can show that for $0 < \alpha \leq 1/2$ and n large enough the asymptotic expected degree distribution $n(d)$ is given by

$$n(d) = \frac{\alpha}{(1 - 2\alpha)n} d^{-1}.$$

The power-law tail of the degree distribution can be obtained from the empirical distribution by a logarithmic binning, as can be seen in Figure 5.

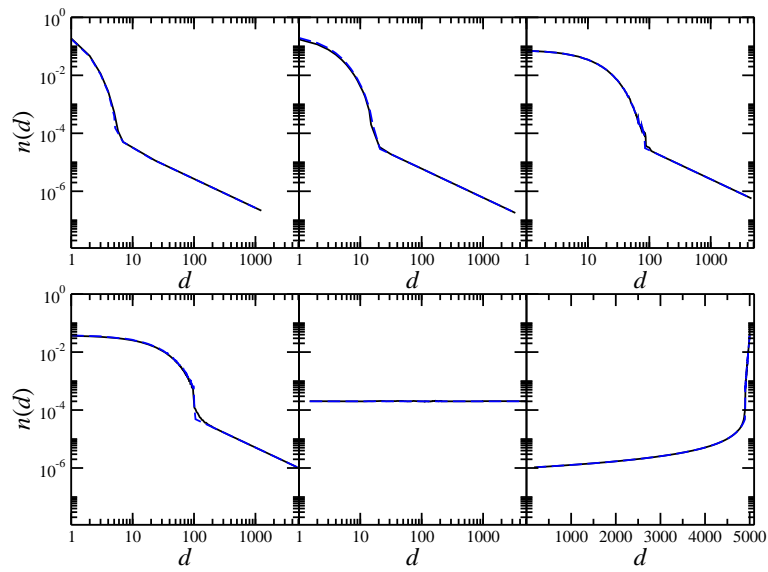


Figure 5: Degree distribution n_d for different values of parameter α and a network size $n = 10\,000$: $\alpha = 0.2$ (top-left plot), $\alpha = 0.4$ (top-center plot), $\alpha = 0.48$ (top-right plot), $\alpha = 0.49$ (bottom-left plot), $\alpha = 0.5$ (bottom-center plot), and $\alpha = 0.52$ (bottom-right plot). The solid line corresponds to the average of simulations while the dashed line indicates the theoretical degree distribution from Proposition 5. The degrees have been binned to smoothen the degree distribution.

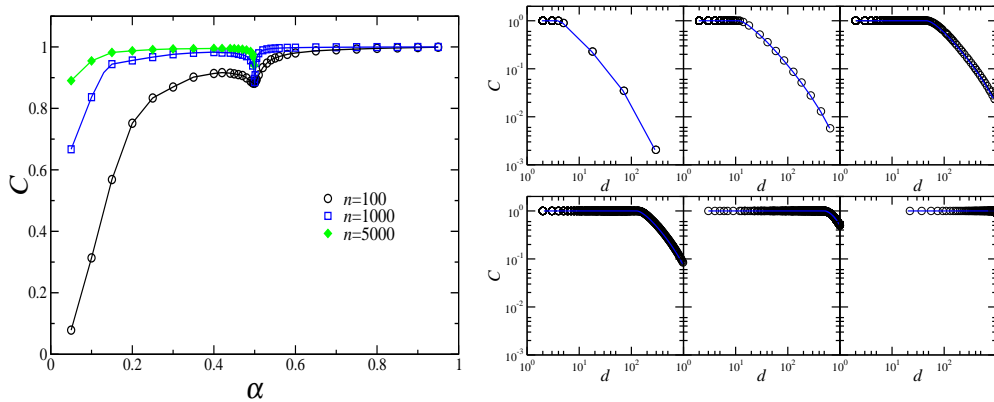


Figure 6: The left panel shows the clustering coefficient C and the right panel the clustering-degree correlation of stationary networks. The symbols correspond to the results obtained by recourse of numerical simulations. The solid lines correspond to the analytical results. We show that the clustering-degree correlation is negative for different values of α and a network size of $n = 1000$. The different plots show different values of α : $\alpha = 0.2$ (top-left plot), $\alpha = 0.4$ (top-center plot), $\alpha = 0.48$ (top-right plot), $\alpha = 0.49$ (bottom-left plot), $\alpha = 0.5$ (bottom-center plot), and $\alpha = 0.52$ (bottom-right plot).

this property in the stationary networks as well, as it is shown in Figure 6 (right).

4.3. Assortativity and Nearest Neighbor Connectivity

We now turn to the study of correlations between the degrees of the agents and their neighbors. This property is usually measured by the network assortativity γ [Newman, 2002, 2003] and nearest neighbor connectivity $d_{nn}(d)$ [Pastor-Satorras et al., 2001]. Dissortative networks are characterized by negative degree correlations between a node and its neighbors and assortative networks show positive degree correlations. In dissortative networks γ is negative and $d_{nn}(d)$ monotonic decreasing while in assortative networks γ is positive and $d_{nn}(d)$ monotonic increasing. We find that in our basic model without capacity constraints (see Appendix C for an extension including capacity constraints in the number of links an agent can maintain) we observe dissortative networks.

Assortativity and neighbor connectivity for different values of the link creation probability α are shown in Figure 7. Clearly, stationary networks are dissortative while the degree of dissortativity decreases with increasing α . However, if we recall the structure of the nested split graphs in Definition 5, to the class the stationary networks belong to, we can see that high degree agents are connected among each other while it is only the low degree agents that are not connected among each other. In this sense agents with high degrees tend to be connected to other agents with high degree. Considering only these agents with high degrees, we can call the network assortative.

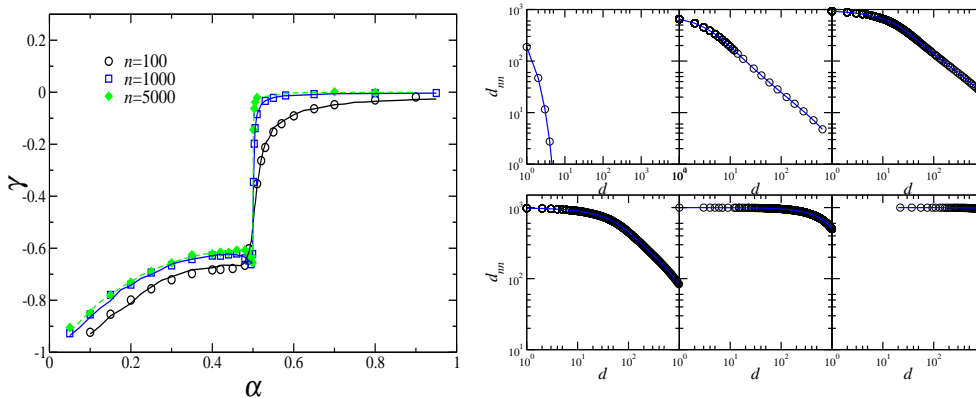


Figure 7: In the left panel we show the assortativity γ of stationary networks. In the right panel we show the average nearest neighbor connectivity d_{nn} for $\alpha = 0.2$ (top-left plot), $\alpha = 0.4$ (top-center plot), $\alpha = 0.48$ (top-right plot), $\alpha = 0.49$ (bottom-left plot), $\alpha = 0.5$ (bottom-center plot), and $\alpha = 0.52$ (bottom-right plot). The symbols correspond to the results obtained by recourse of numerical simulations. The solid lines correspond to the analytical results.

However, the agents with low degrees, that are only connected to agents with high degrees but are disconnected to agents with low degree, are so numerous in the stationary network (for low values of α) that we obtain an overall negative value for the assortativity of the network.

The dissortativity of stationary networks simply reflects the fact that stationary networks are strongly centralized for values of α below $1/2$. As an example consider a star $K_{1,n-1}$. $K_{1,n-1}$ is completely dissortative with $\gamma = -1$. Peripheral agents all have minimum degree one and are only connected to the central agent with maximum degree while the central agent is only connected to the agents with minimum degree. In this sense the dissortativity is simply a measure of centralization in the network.

4.4. Characteristic Path Length

Figure 8 shows the characteristic path length \mathcal{L} and the network efficiency \mathcal{E} (defined in Section B.1.4 in Appendix B). From these figures one can see that the characteristic path length \mathcal{L} never exceeds a distance of two. This means that for all parameter values of α stationary networks are characterized by short distances between agents. Together with the high clustering shown in this section the stationary networks can be seen as “small worlds” [Watts and Strogatz, 1998]. Stationary networks are efficient for values of α larger than $1/2$, in terms of short average distance between agents, while for values of α smaller than $1/2$ they are not. However, this short average distance is attained at the expense of a large number of links.

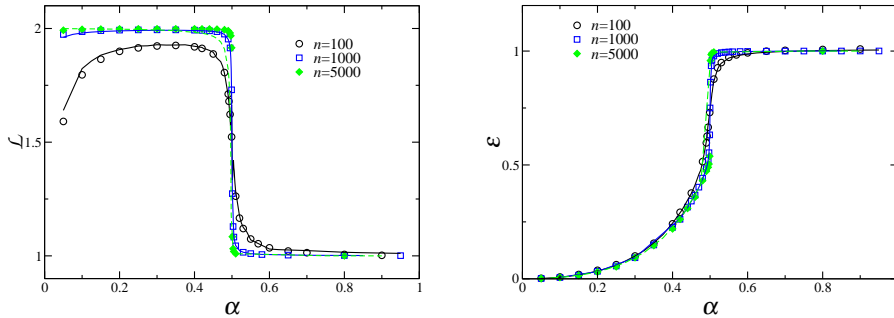


Figure 8: The left panel shows the characteristic path length \mathcal{L} of stationary networks and the right panel shows the results for the network efficiency \mathcal{E} , obtained by recourse of numerical simulations (symbols) and respecting theoretical predictions (lines) of the model.

4.5. Centrality and Centralization in Stationary Networks

In the following section, we analyze the degree of centralization in stationary networks. As we will show, there exists a sharp transition in the centralization as a function of the link creation probability α . This means that stationary networks are either strongly centralized and hierarchical or decentralized and homogeneous, depending on α . In Section 5, we will also find such a transition in the aggregate payoffs and effort levels of the agents.

We use the centralization index introduced by Freeman [1978]. The centralization of a network $G = (N, L)$ is given by

$$\mathcal{C} = \frac{\sum_{u \in N} (\mathcal{C}(u^*) - \mathcal{C}(u))}{\max_{G'} \sum_{v \in N'} (\mathcal{C}(v^*) - \mathcal{C}(v))}, \quad (11)$$

where u^* and v^* are the agents with the highest values of centrality in the current network and the maximum in the denominator is computed over all networks $G' = (N, L')$ with the same number of agents. For the degree, closeness, betweenness and eigenvector centrality measures one obtains the following indices²⁶

$$\begin{aligned} \mathcal{C}_d &= \frac{\sum_{u \in V} (\mathcal{C}_d(u^*) - \mathcal{C}_d(u))}{n^2 - 3n + 2}, \\ \mathcal{C}_c &= \frac{\sum_{u \in V} (\mathcal{C}_c(u^*) - \mathcal{C}_c(u))}{(n^2 - 3n + 2)/(2n - 3)}, \\ \mathcal{C}_b &= \frac{\sum_{u \in V} (\mathcal{C}_b(u^*) - \mathcal{C}_b(u))}{n^3 - 4n^2 + 5n - 2}, \\ \mathcal{C}_v &= \frac{\sum_{u \in V} (\mathcal{C}_v(u^*) - \mathcal{C}_v(u))}{\sqrt{(n-1)/2(\sqrt{n-1}-1)}}. \end{aligned} \quad (12)$$

²⁶For the normalization of all the centralization indices we have used the star $K_{1,n-1}$. For degree, closeness and betweenness centralization it can be shown that the star is the network that maximizes the sum of differences in centrality [Freeman, 1978]. For consistency we also take the star as reference network when computing the eigenvector centralization.

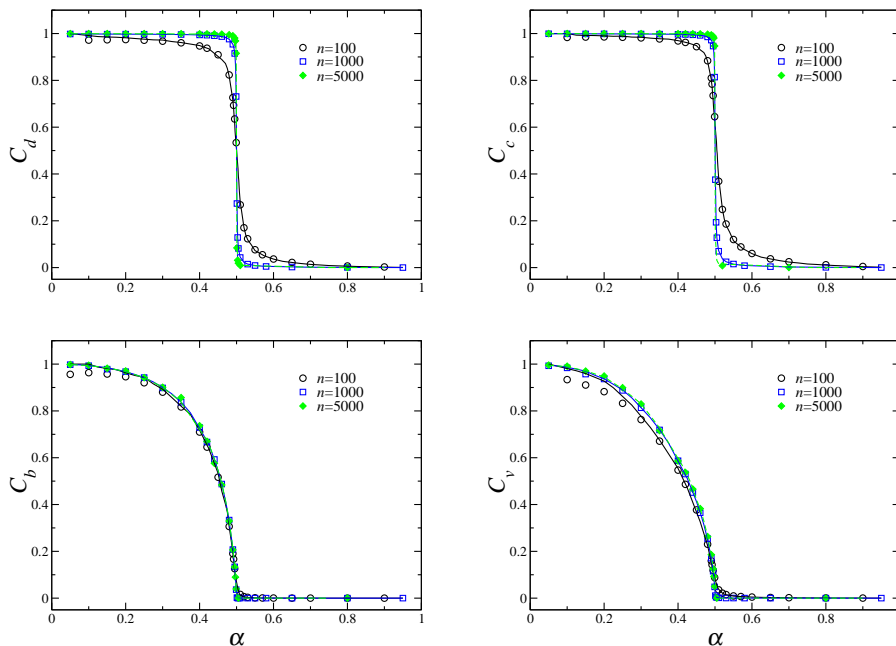


Figure 9: Degree, closeness, betweenness and eigenvector centralization in the stationary networks for different values of α . For all centralization measures we obtain a sharp transition between strongly centralized networks for lower values of α and decentralized networks for higher values of α . Note that we have only considered the connected component for the computation of the different centralization measures.

From Figure 9, showing degree, closeness, betweenness and eigenvector centralization, we clearly see that there exists a phase transition at $\alpha = 1/2$ from highly centralized to highly decentralized networks. This means that for low arrival rates of linking opportunities α (and a strong link decay) the stationary network is strongly polarized, composed mainly of a star (or an inter-linked star as in Goyal and Joshi [2003]), while for high arrival rates of linking opportunities (and a weak link decay) stationary networks are largely homogeneous. We can also see that the transition between these states is sharp.

Our findings are in line with previous works studying the optimal internal communication structure of organizations [Guimerà et al., 2002]. Other works [Calvó-Armengol and Martí, 2009; Dodds et al., 2003; Dupouet and Yildizoglu, 2006; Huberman and Hogg, 1995] have discussed the conditions under which informal organizational networks outperform centralized structures in complex, changing environments and under which conditions hierarchies are more efficient. Similar to Arenas et al. [2008] and Ehrhardt et al. [2006a], we find sharp transitions between largely homogeneous and centralized networks. Moreover, the stationary networks in our model are polarized and

strongly centralized for a low volatility in the environment associated with many linking opportunities whereas they are homogeneous and largely decentralized for a highly volatile environment with few linking opportunities and a strong link decay. The hierarchical structure of stationary networks and its dependency on the volatility is similar to the findings for optimal networks in [Arenas et al. \[2008\]](#).

5. Stationary Networks: Efficiency

We now turn to the investigation of the optimality and efficiency of stationary networks. Following [Jackson and Wolinsky \[1996\]](#) and [Jackson \[2008\]](#), we define the social welfare as the sum of the agents' individual payoffs

$$\Pi(\mathbf{x}^*, G) = \sum_{i=1}^n \pi_i(\mathbf{x}^*, G). \quad (13)$$

We are interested in the solution of the following social planner's problem. Let $\mathcal{G}(n)$ denote the set of connected graphs having n agents in total. The social planner's solution is given by

$$G^* = \operatorname{argmax}_{G \in \mathcal{G}(n)} \Pi(\mathbf{x}^*, G). \quad (14)$$

A graph G^* solving the maximization problem in equation (14) will be denoted as "efficient". The efficient network has been derived in [Ballester et al. \[2006\]](#) and we state their result in the following proposition.

Proposition 7 ([Ballester et al. \[2006\]](#)). *Let $\mathcal{G}(n)$ denote the set of connected graphs having n agents and consider $G \in \mathcal{G}(n)$. Then the efficient network G^* maximizing aggregate equilibrium contribution and payoff is the complete graph K_n .*

This proposition is a direct consequence of Theorem 2 in [Ballester et al. \[2006\]](#) where more links is always better. Moreover, [Corbo et al. \[2006\]](#) have shown that, in the case of strong complementarities, when λ approaches $1/\lambda_{\text{PF}}(G)$, maximizing aggregate equilibrium payoffs is equivalent to maximizing the largest real eigenvalue $\lambda_{\text{PF}}(G)$ of the network G .²⁷

Proposition 8 ([Corbo et al. \[2006\]](#)). *Let $\mathcal{G}(n, m)$ denote the set of connected graphs having n agents and m links and consider $G \in \mathcal{G}(n, m)$. As*

²⁷Following Proposition 7, adding a link always improves aggregate payoff of a network. Through the addition of links we can always make a network connected and therewith increase its aggregate equilibrium payoff. Thus, we restrict our analysis to connected networks.

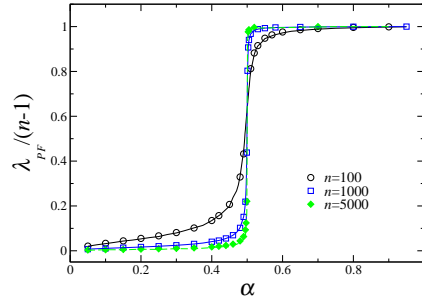


Figure 10: We show the largest eigenvalue of the adjacency matrix of the stationary network relative to the eigenvalue to the complete graph, which is the efficient network, obtained by recourse of numerical simulations (symbols) and respecting theoretical predictions (lines) of the model for different values of α and $n = 200$ agents. For higher values of α the stationary network comes close to the efficient graph which has a largest real eigenvalue of $n - 1$.

$\lambda \uparrow 1/\lambda_{PF}(G)$, *maximizing aggregate equilibrium contribution and payoff reduces to*

$$\max\{\lambda_{PF}(G) : G \in \mathcal{G}(n, m)\}.$$

Proposition 8 tells us that, if we want to compare aggregate payoffs of any two networks G_1 and G_2 , we can compare their largest real eigenvalues, $\lambda_{PF}(G_1)$ and $\lambda_{PF}(G_2)$, in the case of strong complementarities λ . Moreover, from Proposition 7 we know that aggregate payoff is highest in the complete network K_n . K_n also has the highest possible largest real eigenvalue, namely $\lambda_{PF}(K_n) = n - 1$ [Cvetkovic et al., 1997]. Thus, the closer is the largest real eigenvalue $\lambda_{PF}(G(t))$ of a network $G(t)$ to the one of the complete network, the closer it comes to being efficient. Following these observations we show the ratio of the largest real eigenvalue of stationary networks to $n - 1$ for different values of α . We find that for values of α below $1/2$, stationary networks are highly inefficient and a sharp transition occurs for increasing values of α above $1/2$. It is also seen that the transition becomes sharper the larger the network is. This implies that a highly volatile environment and the strong competition of the agents for becoming a hub induces highly inefficient network structures.²⁸

In Appendix C, we introduce capacity constraints and allow for non-local search for new contacts in the link formation process we have discussed so far. We analyze the efficiency of the networks that arise under this extension. We show that, with respect to efficiency, stationary networks in the extended

²⁸It can be shown that the largest real eigenvalue can be increased by concentrating all the links in a densely connected core (clique) for fixed values of the number of links m and nodes n [Cvetkovic et al., 1997].

link formation process show qualitatively the same properties. However, they can differ in their topological properties.

6. Summary and Robustness Analysis

6.1. Summary

In this paper, we develop a two-stage game. In the first stage, agents face a linear-quadratic payoff function that allows for positive utility interdependence between agents. The Nash equilibrium strategies of this game are proportional to the Bonacich centrality of the agents in the network. In the second stage of the game, links are formed as a best response to the current network. More precisely, we introduce a network formation process in which link creation and removal are based on the position of the agents in the network as measured by their Bonacich centrality. Agents only have local information when forming their links and their connections are exposed to a volatile environment. We show that the emerging stationary networks fall into a well defined class of graphs called nested-split graphs. Moreover, these networks exhibit empirically observed properties of social and economic networks. We also find that there exists a sharp transition in the network density from highly centralized to decentralized networks. A similar transition can be observed in the efficiency of stationary networks.

6.2. Robustness Analysis

We discuss different generalizations of our model. First, our analysis is restricted to linear-quadratic utility function (see Equation (1)), capturing linear externalities in players' efforts. This leads to a Nash-equilibrium payoff which is a function of the Bonacich centrality of each player (see Equation (5)). For this equilibrium to be characterized, we also impose that λ , the size of the interactions, has to be strictly lower than the inverse of the largest eigenvalue of the adjacency matrix of the network (see Theorem 1). We can generalize our analysis as follows. Consider now a game where players can only form or sever links but do not choose effort levels. In that case, if we use as payoffs Equation (5) or any increasing transformation of this payoff, then, by considering the network formation process defined in Definition 3, all our results will be the same without, however, relying on any specific form of the utility function. The only requirement is that the utility of each player is increasing in her Bonacich centrality. We can go even further. Consider again the game where players do not choose efforts, then all our results will be valid if the utility function of each player is an increasing function of her closeness centrality²⁹ or given by the utility function of the *connections model* of Jackson and Wolinsky [1996] when costs

²⁹Observe that for degree centrality, a player is indifferent between creating a link with anybody in the network, because any link will increase her degree by one. In that case,

are zero.³⁰ Furthermore, if the utility function of a player is increasing in the number of links of her direct neighbors or any centrality measure³¹ of her direct neighbors (see Corollary 11), then all results are also valid. Observe that in all the cases when we do not use the Bonacich centrality in the utility function, then not only we do not rely on any specific form of the utility function but we do not even need the eigenvalue condition mentioned above.

Second, in our network-formation game defined in Definitions 2 and 3, we impose that, when forming a link (a) a player i needs to choose only among her second-order neighbors, (b) player i has to be the best-response for the chosen second-order neighbor j . Because the networks that emerge are always nested-split graphs, these two assumptions turn out not to be necessary. Indeed, because of the specificity of nested-split graphs, where the maximum distance between players is 2, all the possible players are already contained in the second-order neighbors. So assumption (a) is not necessary. Also, when player i has the possibility to create a link with j , the latter will always accept because it increases her payoff. In other words, i does not need to be the best response for j to increase her payoff and, as a result, (b) is not needed. It turns out, however, that in nested-split graphs, i is always the best response for j . This is a result of the network formation game and not an assumption.

Third, we have considered in this paper a network formation game with perfect information. Some researchers have criticized the fact that, in such models, we are demanding too much from the agents since they need to know all the network structure to make a decision. Because the networks that emerge are always nested-split graphs, which implies in particular that the distance between two agents is at most 2, this assumption is not very strong in our framework. Indeed, we are only imposing that individuals know the decision of the friends of their friends.

Fourth, in Section 4.1, for the nodes in the dominating sets, we obtain a power-law degree distribution with exponent minus one. We can extend our model to obtain a degree distribution with an arbitrary power law tail by making the probability of creating a link for player i depending on, $|D_i|$, the size of the degree partition she belongs to.

Finally, with our network formation game, we always obtain negative degree-degree correlations (i.e. our networks are dissortative). In Appendix C, we extend our game by including capacity constraints in the number of

we could impose some condition to guarantee that she will connect to the player with the highest degree, and then our results will hold.

³⁰Observe that, in our model, there are indirect costs because when a player is chosen with probability $1 - \alpha$, she is obliged to sever a link, which is costly.

³¹For betweenness centrality, we would need to impose some condition stating that, when indifferent, a player will always connect to the player with the highest degree.

links an agent can maintain and a global search mechanism for new linking partners. We find that by introducing capacity constraints and global search, stationary networks can become assortative. Thus, we are able to reproduce all topological properties of empirically observed social and economic networks. Moreover, the emergence of assortativity and positive degree-correlations, respectively, can be explained by considering limitations in the number of links an agent can maintain. This may be of particular relevance for social networks and give an explanation for the distinction between assortative social networks and disassortative technological networks suggested in Newman [2002].

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Appendix

A. Network Definitions and Characterizations

A *network* (*graph*) G is the pair (N, L) consisting of a set of agents (*vertices* or *nodes*) $N = \{1, \dots, n\}$ and a set of links L (*edges*) between them. A link ij is *incident* with the vertex $v \in N$ in the network G whenever $i = v$ or $j = v$. There exists a link between vertices i and j such that $a_{ij} = 1$ if $ij \in L$ and $a_{ij} = 0$ if $ij \notin L$. The *neighborhood* of an agent $i \in N$ is the set $\mathcal{N}_i = \{j \in N : ij \in L\}$. The *degree* d_i of an agent $i \in N$ gives the number of links incident to agent i . Clearly, $d_i = |\mathcal{N}_i|$. Let $\mathcal{N}_i^{(2)} = \bigcup_{j \in \mathcal{N}_i} \mathcal{N}_j \setminus (\mathcal{N}_i \cup \{i\})$ denote the second-order neighbors of agent i . Similarly, the k -th order neighborhood of agent i is defined recursively from $\mathcal{N}_i^{(0)} = i$, $\mathcal{N}_i^{(1)} = \mathcal{N}_i$ and

$$\mathcal{N}_i^{(k)} = \bigcup_{j \in \mathcal{N}_i^{(k-1)}} \mathcal{N}_j \setminus \left(\bigcup_{l=0}^{k-1} \mathcal{N}_i^{(l)} \right).$$

A *walk* in G of length k from i to j is a sequence $p = \langle i_0, i_1, \dots, i_k \rangle$ of agents such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and i_p and i_{p+1} are directly linked, for all $0 \leq p \leq k-1$. Agents i and j are said to be *indirectly linked* in G if there exists a walk from i to j in G . An agent $i \in N$ is *isolated* in G if $a_{ij} = 0$ for all j . The network G is said to be *empty* when all its agents are isolated.

A *subgraph*, G' , of G is the graph of subsets of the agents, $N(G') \subseteq N(G)$, and links, $L(G') \subseteq L(G)$. A graph G is *connected*, if there is a path connecting every pair of agents. Otherwise G is disconnected. The *components* of a graph G are the maximally connected subgraphs. A component is said to be *minimally connected* if the removal of any link makes the component disconnected.

A *dominating set* for a graph $G = (N, L)$ is a subset S of N such that every node not in S is connected to at least one member of S by a link. An *independent set* is a set of nodes in a graph in which no two nodes are adjacent. For example the central node in a star $K_{1,n-1}$ forms a dominating set while the peripheral nodes form an independent set.

In a *complete* graph K_n , every agent is adjacent to every other agent. The graph in which no pair of agents is adjacent is the empty graph \overline{K}_n . A *clique* $K_{n'}$, $n' \leq n$, is a complete subgraph of the network G . A graph is *k-regular* if every agent i has the same number of links $d_i = k$ for all $i \in N$. The complete graph K_n is $(n-1)$ -regular. The cycle C_n is 2-regular. In a *bipartite graph* there exists a partition of the agents in two disjoint sets V_1 and V_2 such that each link connects an agent in V_1 to an agent in V_2 . V_1 and V_2 are independent sets with cardinalities n_1 and n_2 , respectively. In a complete bipartite graph K_{n_1, n_2} each agent in V_1 is connected to each other

agent in V_2 . The *star* $K_{1,n-1}$ is a complete bipartite graph in which $n_1 = 1$ and $n_2 = n - 1$.

The *complement* of a graph G is a graph \bar{G} with the same nodes as G such that any two nodes of \bar{G} are adjacent if and only if they are not adjacent in G . For example the complement of the complete graph K_n is the empty graph \bar{K}_n .

Let \mathbf{A} be the symmetric $n \times n$ *adjacency matrix* of the network G . The element $a_{ij} \in \{0, 1\}$ indicates if there exists a link between agents i and j such that $a_{ij} = 1$ if $ij \in L$ and $a_{ij} = 0$ if $ij \notin L$. The k -th power of the adjacency matrix is related to walks of length k in the graph. In particular, $(\mathbf{A}^k)_{ij}$ gives the number of walks of length k from agent i to agent j . The *eigenvalues* of the adjacency matrix \mathbf{A} are the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ has a nonzero solution vector \mathbf{v}_i , which is an *eigenvector* associated with λ_i for $i = 1, \dots, n$. Since the adjacency matrix \mathbf{A} of an undirected graph G is real and symmetric, the eigenvalues of \mathbf{A} are real, $\lambda_i \in \mathbb{R}$ for all $i = 1, \dots, n$. Moreover, if \mathbf{v}_i and \mathbf{v}_j are eigenvectors for different eigenvalues, $\lambda_i \neq \lambda_j$, then \mathbf{v}_i and \mathbf{v}_j are orthogonal, i.e. $\mathbf{v}_i^T \mathbf{v}_j = 0$ if $i \neq j$. In particular, \mathbb{R}^n has an orthonormal basis consisting of eigenvectors of \mathbf{A} . Since \mathbf{A} is a real symmetric matrix, there exists an orthogonal matrix \mathbf{S} such that $\mathbf{S}^T \mathbf{A} \mathbf{S} = \mathbf{D}$ (that is $\mathbf{S}^T = \mathbf{S}^{-1}$) and $\mathbf{S}^T \mathbf{A} \mathbf{S} = \mathbf{D}$, where \mathbf{D} is the diagonal matrix of eigenvalues of \mathbf{A} and the columns of \mathbf{S} are the corresponding eigenvectors. The *Perron-Frobenius eigenvalue* $\lambda_{\text{PF}}(G)$ is the *largest real eigenvalue* of \mathbf{A} associated with G , i.e. all eigenvalues λ_i of \mathbf{A} satisfy $|\lambda_i| \leq \lambda_{\text{PF}}(G)$ for $i = 1, \dots, n$ and there exists an associated nonnegative eigenvector $\mathbf{v}_{\text{PF}} \geq 0$ such that $\mathbf{A}\mathbf{v}_{\text{PF}} = \lambda_{\text{PF}}(G)\mathbf{v}_{\text{PF}}$. For a connected graph G the adjacency matrix \mathbf{A} has a unique largest real eigenvalue $\lambda_{\text{PF}}(G)$ and a positive associated eigenvector $\mathbf{v}_{\text{PF}} > 0$. There exists a relation between the number of walks in a graph and its eigenvalues. The number of closed walks of length k from a agent i in G to herself is given by $(\mathbf{A}^k)_{ii}$ and the total number of closed walks of length k in G is $\text{tr}(\mathbf{A}^k) = \sum_{i=1}^n (\mathbf{A}^k)_{ii} = \sum_{i=1}^n \lambda_i^k$. We further have that $\text{tr}(\mathbf{A}) = 0$, $\text{tr}(\mathbf{A}^2)$ gives twice the number of links in G and $\text{tr}(\mathbf{A}^3)$ gives six times the number of triangles in G .

B. Topological Properties of Nested Split Graphs

In this Appendix we discuss in more detail the topological properties of nested split graphs that arise from our network formation process. We first derive several network statistics for nested split graphs. We compute the degree distribution, the clustering coefficient, average nearest neighbor connectivity and the characteristic path length in a nested split graph. In particular, we show that connected nested split graphs have small characteristic path length, which is at most two. We then analyze different

measures of centrality in a nested split graph.³² From the expressions of these centrality measures we then can show that degree, closeness, eigenvector and Bonacich centrality induce the same ordering of nodes in a nested split graph. If the ordering is not strict, then this holds also for betweenness centrality. As we elaborate in more detail in Section 6.2 this has important implications for the generality of our model.

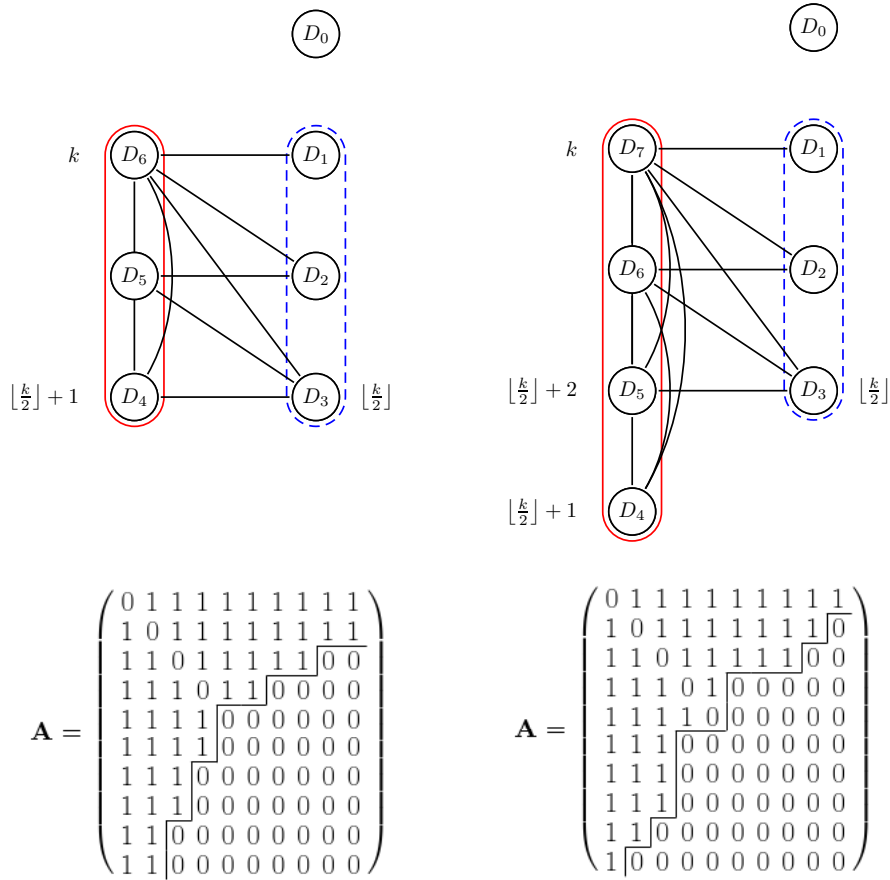


Figure 11: Representation of nested split graphs and their degree partitions \mathbf{D} (top) with corresponding adjacency matrices \mathbf{A} (bottom). A line between D_i and D_j indicates that every node in D_i is adjacent to every node in D_j . The partitions included in the solid frame (D_i with $\lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k$) are the dominating sets while the partitions in the dashed frame (D_i with $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$) are the independent sets. The figure at the top left considers the case of $k = 6$ (even) and the figure at the top right the case of $k = 7$ (odd). The illustration follows Mahadev and Peled [1995, p. 11].

³²See Wasserman and Faust [1994] for an overview of different measures of centrality.

B.1. Network Statistics

In the following sections we will compute the degree connectivity, the clustering coefficient, assortativity and average nearest neighbor connectivity and the characteristic path length in a nested split graph G as a function of the degree partition \mathbf{D} (introduced in Definition 4).

B.1.1. Degree Connectivity

The nested neighborhood structure of a nested split graph allows us to compute the degrees of the nodes according to a recursive equation that is stated in the next corollary.

Corollary 3. *Consider a nested split graph $G = (N, L)$ and let $\mathbf{D} = (D_0, D_1, \dots, D_k)$ be the degree partition of G . Then $d_u = 0$ if $u \in D_0$ and for each $u \in D_i$, $v \in D_{i-1}$, $i = 1, \dots, k$, we get*

$$d_u = \begin{cases} d_v + |D_{k-i+1}|, & \text{if } i \neq \lfloor \frac{k}{2} \rfloor + 1, \\ d_v + |D_{k-i+1}| - 1, & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \end{cases} \quad (15)$$

or equivalently

$$d_u = \begin{cases} \sum_{j=1}^i |D_{k+1-j}|, & \text{if } 1 \leq i \leq \lfloor \frac{k}{2} \rfloor, \\ \sum_{j=1}^i |D_{k+1-j}| - 1, & \text{if } \lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k. \end{cases} \quad (16)$$

Equation (15) shows that the neighborhoods of the agents in a nested split graph are nested (see also Definition 5). The degrees of the agents in ascending order of the graph in Figure 11, top left, are 2, 3, 4, 5, 7, 9 while in the graph in Figure 11, top right, they are 1, 2, 3, 4, 7, 8, 9.

B.1.2. Clustering Coefficient

The clustering coefficient $\mathcal{C}(u)$ for an agent u is the proportion of links between the agents within her neighborhood \mathcal{N}_u divided by the number of links that could possibly exist between them [Watts and Strogatz, 1998]. It is given by

$$\mathcal{C}(u) = \frac{|\{vw : v, w \in \mathcal{N}_u \wedge vw \in L\}|}{d_u(d_u - 1)/2}. \quad (17)$$

In a nested split graph the clustering coefficient can be derived from the degree partition, as the following corollary shows.

Corollary 4. *Consider a nested split graph $G = (N, L)$ and let $\mathbf{D} = (D_0, D_1, \dots, D_k)$ be the degree partition of G . Denote by $S_D^i = \sum_{j=i}^k |D_j|$. Then for each*

$u \in D_i$, $i = 0, \dots, k$, and $d_u \geq 2$, the clustering coefficient is given by

$$\mathcal{C}(u) = \begin{cases} 0, & \text{if } i = 0, \\ 1, & \text{if } 1 \leq i \leq \lfloor \frac{k}{2} \rfloor, \\ \frac{1}{d_u(d_u-1)} \left(S_D^{\lfloor \frac{k}{2} \rfloor + 1} - 1 \right) \left[\left(S_D^{\lfloor \frac{k}{2} \rfloor + 1} - 2 \right) + \right. \\ \left. 2|D_{\lfloor \frac{k}{2} \rfloor}| \right], & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \quad k \text{ even}, \\ \frac{1}{d_u(d_u-1)} \left(S_D^{\lfloor \frac{k}{2} \rfloor + 1} - 1 \right) \left(S_D^{\lfloor \frac{k}{2} \rfloor + 1} - 2 \right), & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \quad k \text{ odd}, \\ \frac{1}{d_u(d_u-1)} \left[\left(S_D^{\lfloor \frac{k}{2} \rfloor + 1} - 1 \right) \left(S_D^{\lfloor \frac{k}{2} \rfloor + 1} - 2 \right) + \right. \\ \left. 2 \sum_{j=k-i+1}^{\lfloor \frac{k}{2} \rfloor} |D_j| \left(S_D^{k-j+1} - 1 \right) \right], & \text{if } \lfloor \frac{k}{2} \rfloor + 2 < i \leq k, \end{cases} \quad (18)$$

where d_u is given by Equation (16).

PROOF OF COROLLARY 4. Note that for all agents in the independent sets, $u \in D_i$ with $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, the clustering coefficient is one, since their neighbors are all connected among each other. Next, we consider the agents $u \in D_i$ with $\lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k$ and degree $d_u = \sum_{j=1}^i |D_{k+1-j}| - 1$. The neighbors of agent u in the dominating sets are all connected among each other with a total of $\frac{1}{2} \left(\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| - 1 \right) \left(\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| - 2 \right)$ links, excluding agent u from the dominating set. The neighbors of u in the independent sets are not connected. Finally, we consider the links between neighbors for which one neighbor is in a dominating set and one neighbor is in an independent set. If k is even we get $\sum_{j=k-i+1}^{\lfloor \frac{k}{2} \rfloor} |D_j| \left(\sum_{l=k-j+1}^k |D_l| - 1 \right)$ links, excluding agent u in the dominating set (see Figure 11 (left)). If k is odd there is no such contribution for the agents in the set $D_{\lfloor \frac{k}{2} \rfloor + 1}$ (see Figure 11 (right)). Putting these contributions together we obtain the clustering coefficient of an agent $u \in D_i$ for all $i = 1, \dots, k$, as given by Equation (18). \square

The total clustering coefficient is the average of the clustering coefficients over all agents,

$$\mathcal{C} = \frac{1}{n} \sum_{u \in N} \mathcal{C}(u). \quad (19)$$

The clustering coefficients of the agents in ascending order of the graph in Figure 11, top left, are $5/12, 5/12, 13/21, 9/10, 1, 1, 1, 1, 1, 1$, with a total clustering coefficient of $\mathcal{C} = 0.84$. In the graph in Figure 11, top right, it is $13/36, 13/28, 4/7, 1, 1, 1, 1, 1, 1, 1$, with a total clustering of $\mathcal{C} = 0.74$.

B.1.3. Assortativity and Nearest Neighbor Connectivity

There exists a measure of degree correlation called “average nearest neighbor connectivity” [Pastor-Satorras et al., 2001]. More precisely, the average nearest neighbor connectivity $d_{nn}(u)$ is the average degree of the neighbors of an agent with degree d_u . It is defined by

$$d_{nn}(u) = \frac{1}{d_u} \sum_{v \in \mathcal{N}_u} d_v. \quad (20)$$

In a nested split graph the average nearest neighbor connectivity is determined by its degree partition.

Corollary 5. *Consider a nested split graph $G = (N, L)$ and let $\mathbf{D} = (D_0, D_1, \dots, D_k)$ be the degree partition of G . Denote by $S_D^i = \sum_{j=1}^i |D_{k+1-j}|$. Then for each $u \in D_i$, $i = 0, \dots, k$,*

$$d_{nn}(u) = \begin{cases} 0, & \text{if } i = 0, \\ \frac{1}{S_D^i} \sum_{j=1}^i |D_{k+1-j}| \left(S_D^{k+1-j} - 1 \right), & \text{if } i = 1, \dots, \lfloor \frac{k}{2} \rfloor, \\ \frac{1}{S_D^{\lfloor \frac{k}{2} \rfloor + 1} - 1} \left[\sum_{j=\lfloor \frac{k}{2} \rfloor + 2}^k |D_j| \left(S_D^j - 1 \right) \right. \\ \left. + \left(|D_{\lfloor \frac{k}{2} \rfloor + 1}| - 1 \right) \left(S_D^{\lfloor \frac{k}{2} \rfloor + 1} - 1 \right) + |D_{\lfloor \frac{k}{2} \rfloor}| S_D^{\lfloor \frac{k}{2} \rfloor} \right], & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \quad k \text{ even}, \\ \frac{1}{S_D^{\lfloor \frac{k}{2} \rfloor + 1} - 1} \left[\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| \left(S_D^j - 1 \right) \right] - 1, & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \quad k \text{ odd}, \\ \frac{1}{S_D^i - 1} \left[\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| \left(S_D^j - 1 \right) \right. \\ \left. + \sum_{j=k-i+1}^{\lfloor \frac{k}{2} \rfloor} |D_j| S_D^j \right] - 1, & \text{if } i = \lfloor \frac{k}{2} \rfloor + 2, \dots, k \end{cases} \quad (21)$$

PROOF OF COROLLARY 5. First, consider an agent $u \in D_i$ with $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$ corresponding to the independent sets. We know that the number of neighbors (degree) of agent u is given by $\sum_{j=1}^i |D_{k+1-j}|$. The neighbors of agent u are the agents in the dominating sets with degrees given in Equation (16). Thus, the number of neighbors of the neighbors of u in the sets D_{k+1-j} is $\sum_{l=1}^{k+1-j} |D_{k+1-l}| - 1$. Putting the above results together, we obtain for the average nearest neighbor connectivity of agent $u \in D_i$, $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$, the following expression.

$$d_{nn}(u) = \frac{1}{\sum_{j=1}^i |D_{k+1-j}|} \sum_{j=1}^i |D_{k+1-j}| \left(\sum_{l=1}^{k+1-j} |D_{k+1-l}| - 1 \right). \quad (22)$$

Next, we consider an agent u in the set D_i with $\lfloor \frac{k}{2} \rfloor + 2 \leq i \leq k$ corresponding to the dominating sets. The number of neighbors of agent u is given by

$\sum_{j=1}^i |D_{k+1-j}| - 1$. The number of neighbors of an agent $v \in D_j$, $\lfloor \frac{k}{2} \rfloor + 1 \leq j \leq k$ in the dominating sets is given by $\sum_{l=1}^j |D_{k+1-l}| - 1$. Since agent u is connected to all other agents in the dominating sets, we can sum over all their neighborhoods with a total of $\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| \left(\sum_{l=1}^j |D_{k+1-l}| - 1 \right)$ neighbors. Note however, that we have to subtract agent u herself from this sum. Moreover, the number of neighbors of an agent $w \in D_j$, $1 \leq j \leq \lfloor \frac{k}{2} \rfloor$ in the independent sets is given by $\sum_{l=1}^j |D_{k+1-l}|$. Thus, the average nearest neighbor connectivity of agent $u \in D_i$, $\lfloor \frac{k}{2} \rfloor + 2 \leq i \leq k$, is given by

$$d_{nn}(u) = \frac{1}{\sum_{j=1}^i |D_{k+1-j}| - 1} \left[\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| \left(\sum_{l=1}^j |D_{k+1-l}| - 1 \right) + \sum_{j=k-i+1}^{\lfloor \frac{k}{2} \rfloor} |D_j| \sum_{l=1}^j |D_{k+1-l}| \right] - 1. \quad (23)$$

In a similar way we can consider the cases $i = \lfloor \frac{k}{2} \rfloor + 1$ for both k even and k odd. \square

When the average nearest neighbor connectivity is a monotonic increasing function of the degree d , then the network is assortative, while, if it is monotonic decreasing with d , it is disassortative [Newman, 2002; Pastor-Satorras et al., 2001]. Nested split graphs are disassortative, since for $i < j$ and $d_u \in D_i < d_v \in D_j$ it follows that $d_{nn}(u) > d_{nn}(v)$. This is because the higher is the degree of an agent in a dominating set, the more neighbors she has from the independent sets with low degrees, which decreases her average nearest neighbor connectivity. For example, the average nearest neighbor connectivities of the agents in the graph in Figure 11, top left, in ascending order are 13/3, 13/3, 37/7, 33/5, 15/2, 15/2, 25/3, 25/3, 9, 9 while in the graph in Figure 11, top right, they are 35/9, 35/8, 34/7, 7, 7, 8, 8, 8, 17/2, 9.

B.1.4. Characteristic Path Length

The characteristic path length is defined as the number of links in the shortest path between two agents, averaged over all pairs of agents [Watts and Strogatz, 1998]. This can be written as

$$\mathcal{L} = \frac{1}{n(n-1)/2} \sum_{u \neq v} d(u, v), \quad (24)$$

where $d(u, v)$ is the geodesic (shortest path) between agent u and agent v in $N \setminus D_0$.³³ Then the characteristic path length in a nested split graph is given by the following corollary.

³³Note that we do not consider the isolated agents in the set D_0 because the characteristic path length \mathcal{L} is not defined for disconnected networks.

Corollary 6. Consider a nested split graph $G = (N, L)$ and let $\mathbf{D} = (D_0, D_1, \dots, D_k)$ be the degree partition of G . Then the characteristic path length of G is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{n(n-1)/2} \left[\frac{1}{2} \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| \left(\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| - 1 \right) + \right. \\ &\quad \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} |D_j| \left(\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} |D_j| - 1 \right) + \\ &\quad \left. \sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} |D_l| \left(\sum_{j=k-l+1}^k |D_j| + 2 \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^{k-l} |D_j| \right) \right]. \end{aligned} \quad (25)$$

PROOF OF COROLLARY 6. We first consider all pairs of agents in the dominating sets. All these agents are adjacent to each other and thus the shortest path between them has length one. Moreover, there are $\frac{1}{2} \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| \left(\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| - 1 \right)$ pairs of agents in the dominating sets.

Next, we consider all pairs of agents in the independent sets. From Equation (31) we know that all of them are at a distance of two links separated from each other. Moreover, there are $\frac{1}{2} \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} |D_j| \left(\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} |D_j| - 1 \right)$ pairs of agents in which both agents stem from an independent set.

Finally, we consider the pairs of agents in which one agent is in the independent set D_1 and the other in a dominating set. Then there are $|D_1||D_k|$ pairs of agents with shortest path 1 and $|D_1| \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^{k-1} |D_j|$ pairs of agents with shortest path 2. Similarly, we can consider the pairs in which one agent is in the set D_2 . Then we have $|D_2|(|D_k| + |D_{k-1}|)$ pairs of agents with shortest path 1 and $|D_2| \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^{k-2} |D_j|$ pairs of agents with shortest path 2. Finally, if one agent is in the set $D_{\lfloor \frac{k}{2} \rfloor}$ then we have $|D_{\lfloor \frac{k}{2} \rfloor}| \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j|$ pairs of agents with distance 1 and none with distance 2, if k is even (see Figure 11, top left). If k is odd (see Figure 11, top right), and we have one agent is in the set $D_{\lfloor \frac{k}{2} \rfloor}$ then we have $|D_{\lfloor \frac{k}{2} \rfloor}| \sum_{j=\lfloor \frac{k}{2} \rfloor + 2}^k |D_j|$ pairs of agents with distance 1 and $|D_{\lfloor \frac{k}{2} \rfloor}| |D_{\lfloor \frac{k}{2} \rfloor + 1}|$ pairs with distance 2.

Therefore, the average path length \mathcal{L} defined in Equation (24) is given by the following equation

$$\begin{aligned} \frac{n(n-1)}{2} \mathcal{L} &= \frac{1}{2} \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| \left(\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j| - 1 \right) + \\ &\quad 2 \frac{1}{2} \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} |D_j| \left(\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} |D_j| - 1 \right) + \\ &\quad \sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} |D_l| \left[\sum_{j=k-l+1}^k |D_j| + 2 \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^{k-l} |D_j| \right]. \end{aligned} \quad (26)$$

□

Considering the graph in Figure 11, top left, the characteristic path length is $\mathcal{L} = 22/15$ while in the graph in Figure 11, top right, we get $\mathcal{L} = 68/45$.

Taking the inverse of the shortest path length one can introduce a related measurement, the network efficiency³⁴ \mathcal{E} , that is also applicable to disconnected networks [Latora and Marchiori, 2001]

$$\mathcal{E} = \frac{1}{n(n-1)} \sum_{u \neq v} \frac{1}{d(u,v)}. \quad (27)$$

Finally, we find that in a connected nested split graph agents are at most two links separated from each other and thus these graphs are characterized by a short characteristic path length.

B.2. Centrality

In the next sections we analyze different measures of centrality in a nested split graph G . We derive the expressions for degree, closeness and betweenness centrality as a function of the degree partition of G . Finally, we show that these measures are similar in the sense that they induce the same ordering of the nodes in G based on their centrality values.

B.2.1. Degree Centrality

The degree centrality of an agent $u \in N$ is given by the proportion of agents that are adjacent to u [Wasserman and Faust, 1994]. We obtain the normalized degree centrality simply by dividing the degree of agent u with the maximum degree $n - 1$. This yields the following corollary.

Corollary 7. *Consider a nested split graph $G = (N, L)$ and let $\mathbf{D} = (D_0, D_1, \dots, D_k)$ be the degree partition of G . Then for each $u \in D_i$, $i = 0, \dots, k$, the degree centrality is given by*

$$C_d(u) = \begin{cases} \frac{1}{n-1} \sum_{j=1}^i |D_{k+1-j}|, & \text{if } 1 \leq i \leq \lfloor \frac{k}{2} \rfloor, \\ \frac{1}{n-1} \left(\sum_{j=1}^i |D_{k+1-j}| - 1 \right), & \text{if } \lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k. \end{cases} \quad (28)$$

PROOF OF COROLLARY 7. The result follows directly from Corollary 3. \square

We observe that degree centrality as well as the degree are increasing with increasing index i of the set D_i to which agent u belongs. Degree centralities for the graphs shown in Figure 11 can be derived from the degrees given in Section B.1.1 by dividing the degrees with $n - 1$.

³⁴The network efficiency must not be confused with the efficiency of a network. The first is related to short paths in the network while the latter measures social welfare, that is, the efficient network maximizes aggregate payoff.

B.2.2. Closeness Centrality

Excluding the isolated nodes in G the closeness centrality of agent $u \in N \setminus D_0$ is defined as [Beauchamp, 1965; Sabidussi, 1966]:

$$\mathcal{C}_c(u) = \frac{n-1}{\sum_{v \neq u} d(u, v)}. \quad (29)$$

where $d(u, v)$ measures the shortest path between agent u and agent v in $N \setminus D_0$. For a nested split graph we obtain the following corollary.

Corollary 8. *Consider a nested split graph $G = (N, L)$ and let $\mathbf{D} = (D_0, D_1, \dots, D_k)$ be the degree partition of G . Then for each $u \in D_i$, $i = 0, \dots, k$, the closeness centrality is given by*

$$\mathcal{C}_c(u) = \begin{cases} \frac{n-1}{\sum_{j=k-i+1}^k |D_j| + 2 \sum_{j=1}^{k-i} |D_j| - 2}, & \text{if } 1 \leq i \leq \lfloor \frac{k}{2} \rfloor, \\ \frac{n-1}{\sum_{j=k-i+1}^k |D_j| + 2 \sum_{j=1}^{k-i} |D_j| - 1}, & \text{if } \lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k. \end{cases} \quad (30)$$

PROOF OF COROLLARY 8. For both agents in the independent sets, $u \in D_i$ with $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, and in the dominating sets, $u \in D_i$ with $\lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k$, we can compute the length of the shortest paths as follows:

$$d(u, v) = \begin{cases} 1 & \text{for all } v \in \bigcup_{j=k-i+1}^k D_j, \\ 2 & \text{for all } v \in \bigcup_{j=1}^{k-i} D_j. \end{cases} \quad (31)$$

In order to compute the closeness centrality we have to consider all pairs of agents in the graph and compute the length of the shortest path between them, which is given in Equation (31). We obtain for any agent $u \in D_i$, $i = 1, \dots, k$, the following expression

$$\mathcal{C}_c(u) = \begin{cases} \frac{n-1}{\sum_{j=k-i+1}^k |D_j| + 2 \sum_{j=1}^{k-i} |D_j| - 2}, & \text{if } 1 \leq i \leq \lfloor \frac{k}{2} \rfloor \\ \frac{n-1}{\sum_{j=k-i+1}^k |D_j| + 2 \sum_{j=1}^{k-i} |D_j| - 1}, & \text{if } \lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k. \end{cases} \quad (32)$$

Note that we have subtracted 1 and 2 in the denominator, respectively, since the sums would otherwise include the contribution of agent u herself. \square

We have that closeness centrality is identical for all agents in the same set. Also note that $\mathcal{C}_c(u) = 1$ for $u \in D_k$. Moreover, closeness centrality is increasing with increasing degree. The closeness centralities of the agents in descending order for the graph in Figure 11, top left, are 1, 1, 9/11, 9/13, 9/14, 9/14, 9/15, 9/15, 9/16, 9/16 while in the graph in Figure 11, top right, they are 1, 9/10, 9/11, 9/14, 9/14, 9/15, 9/15, 9/15, 9/16, 9/17.

B.2.3. Betweenness Centrality

Betweenness centrality is defined as [Freeman, 1977]

$$C_b(u) = \sum_{u \neq v \neq w} \frac{g(v, u, w)}{g(v, w)}, \quad (33)$$

where $g(v, w)$ denotes the number of shortest paths from agent v to agent w and $g(v, u, w)$ counts the number of paths from agent v to agent w that pass through agent u .

The betweenness centrality for a nested split graph can be derived from its degree partition as follows.³⁵

Corollary 9. *Consider a nested split graph $G = (N, L)$ and let $\mathbf{D} = (D_0, D_1, \dots, D_k)$ be the degree partition of G . Then $C_b(u) = 0$ if $u \in D_i$, $i = 0, \dots, \lfloor \frac{k}{2} \rfloor$ and for each $u \in D_i$, $v \in D_{i-1}$, $i = \lfloor \frac{k}{2} \rfloor + 1, \dots, k$, the betweenness centrality is given by*

$$C_b(u) = \begin{cases} 0 & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \quad k \text{ odd} \\ \frac{|D_{\lfloor \frac{k}{2} \rfloor}| (|D_{\lfloor \frac{k}{2} \rfloor}| - 1)}{\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j|}, & \text{if } i = \lfloor \frac{k}{2} \rfloor + 1, \quad k \text{ even} \\ C_b(v) + \frac{|D_{k-i+1}| (|D_{k-i+1}| - 1)}{\sum_{j=i}^k |D_j|} \\ + \frac{2|D_{k-i+1}| \sum_{j=k-i+2}^{i-1} |D_j|}{\sum_{j=i}^k |D_j|}, & \text{if } \lfloor \frac{k}{2} \rfloor + 2 \leq i \leq k. \end{cases} \quad (34)$$

PROOF OF COROLLARY 9. In this proof, we follow closely Hagberg et al. [2006]. The agents in the independent sets D_i , $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$ do not lie on any shortest path between two other agents in the network and thus their betweenness centrality vanishes. For the agents in the dominating sets we have that the betweenness centrality of the agent $u \in D_{\lfloor \frac{k}{2} \rfloor + 1}$ vanishes if k is odd and is given by $|D_{\lfloor \frac{k}{2} \rfloor}| (|D_{\lfloor \frac{k}{2} \rfloor}| - 1) / \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^k |D_j|$ if k is even. The latter result is due the shortest path between agents that are both in $D_{\lfloor \frac{k}{2} \rfloor}$. Next, consider an agent $u \in D_i$ and $v \in D_{i-1}$, with $\lfloor \frac{k}{2} \rfloor + 2 \leq i \leq k$. Then the betweenness centrality of agent u is given by the following recursive relationship

$$C_b(v) + \frac{|D_{k-i+1}| (|D_{k-i+1}| - 1)}{\sum_{j=i}^k |D_j|} + \frac{2|D_{k-i+1}| \sum_{j=k-i+2}^{i-1} |D_j|}{\sum_{j=i}^k |D_j|}. \quad (35)$$

The first term in Equation (35) is due to the fact that all shortest paths through lower dominating nodes $v \in D_{i-1}$ have the same length as through

³⁵A similar result can be found in Hagberg et al. [2006].

$u \in D_i$. The second term in Equation (35) represents the contribution of paths between nodes in D_{k-i+1} , divided by the number of shortest path passing through the agents in the dominating sets D_j , $i \leq j \leq k$. The third term in Equation (35) represents all path between an agent in D_{k-i+1} and the other being in D_j , $k-i+2 \leq j \leq i-1$, divided by the number of shortest path passing through the agents in the dominating sets D_j , $i \leq j \leq k$. \square

From Corollary 9, we find that the agents in the independent sets D_i with $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ have vanishing betweenness centrality. From the above equation we also observe that the betweenness centrality is increasing with degree such that the agents in D_k have the highest betweenness centrality, the agents in D_{k-1} the second highest betweenness centrality and so on. Thus, the ordering of betweenness centralities follows the degree ordering for all agents in the dominating sets while the agents in the independent sets have vanishing betweenness centrality. For the betweenness centralities of the agents in the graph in Figure 11, top left, we obtain in descending order $109/6, 109/6, 31/6, 1/2, 0, 0, 0, 0, 0, 0$ while in the graph in Figure 11, top right, they are $28, 12, 6, 0, 0, 0, 0, 0, 0, 0$.

B.2.4. Eigenvector Centrality

There is a central property that holds for nested split graphs in relation to Bonacich centrality, namely that the agents with higher degree also have higher Bonacich centrality. Similar to part (i) of Proposition 1 we can give the following corollary.³⁶

Corollary 10. *Let \mathbf{v} be the eigenvector associated with the largest real eigenvalue $\lambda_{PF}(G)$ of the adjacency matrix \mathbf{A} of a nested split graph $G = (N, L)$. For each $i = 1, \dots, n$, v_i is the eigenvector centrality of agent i . Consider a pair of agents $i, j \in N$. If and only if agent i has a higher degree than agent j then i has a higher eigenvector centrality than j , i.e.*

$$d_i > d_j \Leftrightarrow v_i > v_j.$$

PROOF OF COROLLARY 10. The proof is identical to the proof of part (i) of Proposition 1. \square

B.2.5. Centrality Rankings

Putting together the results for different centrality measures derived in the previous sections, we can make the following observation of the rankings of agents for different centrality measures in a nested split graph.

³⁶A similar result can be found in [Grassi et al., 2007].

Corollary 11. *Consider a nested split graph $G = (N, L)$. Let $\mathcal{C}_d, \mathcal{C}_c, \mathcal{C}_b, \mathcal{C}_v$ denote the degree, closeness, betweenness and eigenvector centrality in G . Then for any $l, m \in \{d, c, v\}$, $l \neq m$ and $i, j \in N$ we have that*

$$\mathcal{C}_l(i) \geq \mathcal{C}_l(j) \Leftrightarrow \mathcal{C}_m(i) \geq \mathcal{C}_m(j), \quad (36)$$

and

$$\mathcal{C}_l(i) \geq \mathcal{C}_l(j) \Rightarrow \mathcal{C}_b(i) \geq \mathcal{C}_b(j). \quad (37)$$

PROOF OF COROLLARY 11. The proof is a direct application of Corollaries 7, 8 9 and Proposition 1. \square

If and only if an agent i has the k -th highest degree centrality then i is the agent with the k -th highest closeness and eigenvector centrality. This result also holds for Bonacich centrality (see Proposition 1). Moreover, if an agent i has the k -th highest degree centrality then it also has the k -th highest betweenness centrality and this also holds for closeness, eigenvector and Bonacich centrality, respectively. The ordering induced by degree, closeness eigenvector and Bonacich centrality coincide and these orderings also apply in a weak sense for betweenness centrality. We discuss in Section 6.2 that this allows us to generalize our model to various other centrality measures beyond Bonacich centrality.

C. Capacity Constraints and Global Search

A natural generalization of the model presented in the paper is to allow for the possibility that agents are not accepting to establish a link from another agent that wants to connect to them. The underlying assumption is that agents face capacity constraints in the number of links they can maintain. Such constraints can arise from a possible information overload and congestion [Arenas et al., 2008; Dodds et al., 2003; Fagiolo, 2005; Guimerà et al., 2002; Huberman and Hogg, 1995]. In addition, we assume that agents are not only searching for new contacts among their neighbors' neighbors but also among all agents in the network. However, agents preferably connect to their neighbors' neighbors and only if this fails they search for new contacts at random. This means that, if capacity constraints prevent an agent from forming a link locally, we assume that she tries to link to an agent out of the whole population of agents at random. This mechanism introduces a global search mechanism in the link formation process (see Marsili et al. [2004]; Vega-Redondo [2006] for a similar approach). We find that by introducing capacity constraints and global search, differently to the model presented in the main text, stationary networks can become assortative. Thus, we are able to reproduce all topological properties of empirically observed social and economic networks. Moreover, the emergence of assortativity and positive degree-correlations respectively can be

explained by considering limitations in the number of links an agent can maintain. This may be of particular relevance for social networks and give an explanation for the distinction between assortative social networks and dissortative technological networks suggested in Newman [2002].

We assume that capacity constraints arise from the fact that an agent can only interact with another agent out of her neighborhood at a time. Each neighbor of an agent requests information with probability β .³⁷ Assuming that information requests are independent, the probability that an agent $j \in N$ with d_j links does not receive any information requests from her neighbors is given by $(1-\beta)^{d_j}$. If an agent does not receive such an information request, she can accept an additional link, otherwise not.

Moreover, we allow for the formation of links between agents that are not connected through a common neighbor. This means that agents search globally for new contacts if they cannot connect to the agent with highest centrality among their neighbors' neighbors. When an agent i is selected, she tries to connect to the agent j with the highest degree in her neighborhood \mathcal{N}_i . However, agent $j \in \mathcal{N}_i$ only accepts the link ij with probability $(1-\beta)^{d_j}$, otherwise agent i selects another agent $k \in N \setminus (\mathcal{N}_i \cup \{i, j\})$ out of the whole population of agents (excluding agents i and j) uniformly at random, and this link also has the same acceptance probability $(1-\beta)^{d_k}$ based on the degree of agent k .

In the following, we make a rather technical assumptions. First, in the model exposed in the text, the Bonacich centrality of an agent increases the most if she forms a link to the agent with the highest degree. For the current model we will assume that this property is still approximately true. In most cases this approximation can be made albeit there exist exceptions in which the degree and Bonacich centrality ranking do not coincide [Grassi et al., 2007].

Taking into account the above mentioned capacity constraints in the number of links an agent can form and the possibility to form links outside the second order neighborhood, we generalize the link formation process $(G(t))_{t=0}^{\infty}$ introduced in Section 2.2 as follows:

Definition 7. *We define the network formation process $(G'(t))_{t=0}^{\infty}$, $G'(t) = (N, L'(t))$, as a sequence of networks $G'(0), G'(1), G'(2), \dots$ in which at every step $t = 0, 1, 2, \dots$, an agent $i \in N$ is uniformly selected at random. Then one of the following two events occurs:*

- (i) *With probability $\alpha \in (0, 1)$ agent i receives the opportunity to create an additional link. Let j be the agent in $\mathcal{N}_i^{(2)}$ with the highest degree, that is $d_j \geq d_k$ for all $j, k \in \mathcal{N}_i^{(2)}$. Then with probability $(1-\beta)^{d_j}$ the link ij is formed. Otherwise agent i connects to a randomly selected agent*

³⁷If an agent has to process such a request, she cannot accept an additional link.

- $k \in N \setminus (\mathcal{N}_i \cup \{i, j\})$ with probability $(1 - (1 - \beta)^{d_j}) (1 - \beta)^{d_k}$. If agent i is already connected to all other agents then nothing happens.
- (ii) With probability $1 - \alpha$, the link to the agent j in \mathcal{N}_i with the smallest degree $d_j \leq d_k$ for all $j, k \in \mathcal{N}_i$, decays. If agent i does not have any links then nothing happens.

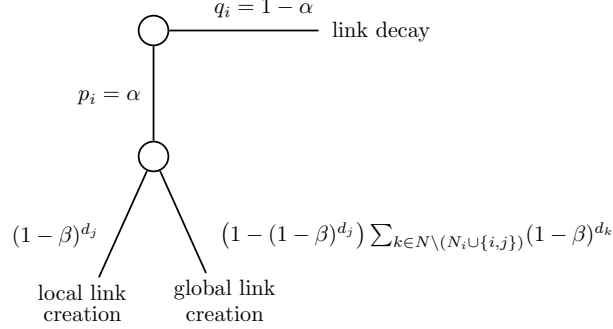


Figure 12: Probabilities with which a randomly selected agent i creates a link and a link of agent i decays, respectively, when capacity constraints are taken into account (assuming that the agent is neither isolated nor fully connected).

An illustration of the above link formation process $(G'(t))_{t=0}^{\infty}$ is shown in Figure 12. An agent i is selected at random either creates a link or the link to the neighbor with lowest centrality decays with probability $q_i = 1 - \alpha$. However, with probability $p_i = \alpha$ agent i is selected to create a link. In this case, agent i forms the link to agent j with highest centrality among her second order neighbors with probability $(1 - \beta)^{d_j}$ and to another agent out of the whole population of agents at random with probability $(1 - (1 - \beta)^{d_j}) \sum_{k \in N \setminus (\mathcal{N}_i \cup \{i, j\})} (1 - \beta)^{d_k}$.

Having introduced the extended network formation process $(G'(t))_{t=0}^{\infty}$ we now investigate its properties by means of computer simulations for values of $\alpha \in [0.2, 0.5]$ and $\beta \in [0.01, 1]$.³⁸ We consider a set of $n = 200$ agents and use a sample of 30 to 40 simulation runs from which we compute the average as an approximation to the stationary network.

Figure 13 shows the clustering and assortativity of stationary networks for different values of α and β . We find that for values of β around 0.1 and in $\alpha \in [0.45, 0.5]$ stationary networks are assortative while displaying a high clustering (albeit lower than in the basic model without capacity

³⁸In the simulation results shown in this section, we assume that if an agent is not free to accept an additional link, (or the agent that is the target of the link can not form an additional link) another agent is selected, until a link is formed. In this way, the values of α in the generalized model are comparable with the basic model without capacity constraints in which α is a measure of the network density.

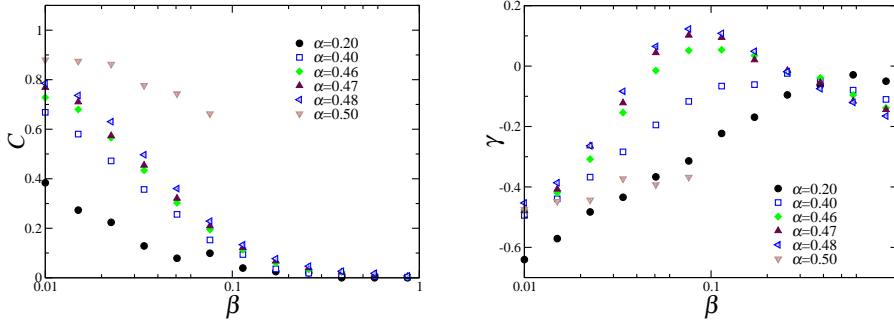


Figure 13: In the left panel we show the clustering coefficient obtained by recourse of numerical simulations of the extended model with capacity constraints for different values of α and β in a network with $n = 200$ agents. In the right panel we show the corresponding network assortativity. Each different curve corresponds to a different value of α . Only agents that are not isolated are considered.

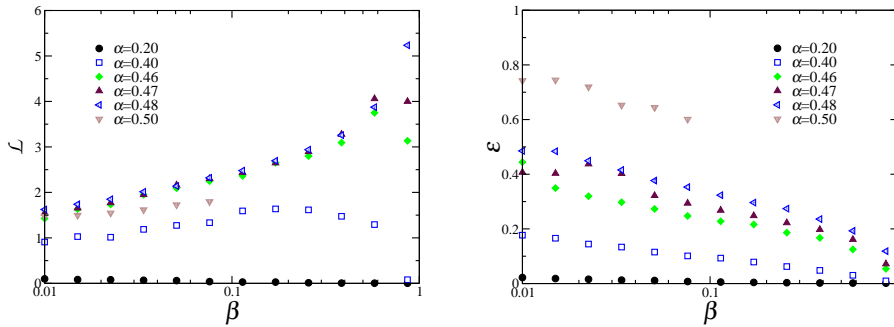


Figure 14: The left panel shows the characteristic path length \mathcal{L} of the stationary network and the right panel shows the results for the network efficiency \mathcal{E} by recourse of numerical simulations of the model with capacity constraints for different values of α and β in a system comprised of $n = 200$ agents.

constraints). In Figure 14 we show the characteristic path length \mathcal{L} and the efficiency \mathcal{E} in terms of short connections in the network. The plots indicate that stationary networks in the extended model exhibit short path lengths between the agents. However, we find that the stationary network may not just consist of one connected component and possibly isolated agents but it may have multiple components. However, there exists a giant component encompassing at least 90% of the agents in all the simulations we studied. We can further analyze the degree distribution of stationary networks and we find that it is highly skewed following an exponential function.

Moreover, we find that the results for different centralization measures show a similar behavior as we have seen already in Section 4.5. There exists a sharp, albeit less pronounced, transition from highly centralized networks

to homogeneous networks by increasing α above $1/2$.

In Figure 15 we show the fraction of the largest real eigenvalue (as a measure of efficiency) of the stationary network compared to the corresponding value of the complete network. The figure resembles the findings in Section 5. For values of $\alpha < 1/2$ stationary networks are highly inefficient with respect to the complete network.

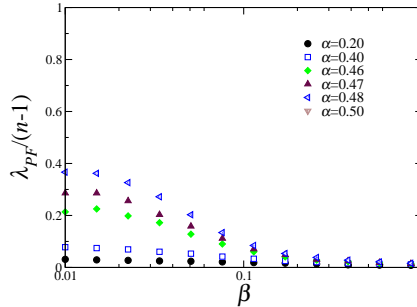


Figure 15: We show the largest eigenvalue of the adjacency matrix normalized to the largest one in a complete graph (which is the efficient network), obtained by recourse of numerical simulations for the model with capacity constraints for different values of α and β in a system comprised of $n = 200$ agents.

In this section we have studied different network statistics for different values of α and β . We find that, by introducing capacity constraints and global search, stationary networks become assortative while exhibiting an exponential degree distribution, high clustering, short average path length and negative clustering-degree correlation. These characteristics can be found in social and economic networks as well. Thus, our model is able to reproduce characteristics of real world networks to the whole extent, ranging from assortative to disassortative networks.

Our findings have an implication for the distinction between assortative and disassortative networks in the literature. As we have discussed in the main text, our network formation process generates stationary networks that are characterized by negative degree-degree correlation and disassortativity. On the other hand, capacity constraints transform stationary networks to exhibiting positive degree-degree correlations and assortativity. This effect may shed some light on the origin of the distinction between technological and social networks suggested in Newman [2002, 2003] where technological networks are characterized by disassortativity and social networks by assortativity. Following our findings, technological networks are facing capacity constraints to a much lower extent than social networks. Consider for example the internet as a technological network and the email network in an organization as a prototype of a social network. The number of hyper-links a website can contain may not be limited as much as the number of social

contacts (measured e.g. by mutual email exchange) an individual in an organization may keep. Thus, the distinction between technological and social networks and the degree of assortativity and degree-degree correlations can be derived from the severity of capacity constraints imposed on the number of links an agent can maintain.

D. Proofs of Propositions, Corollaries and Lemmas

In this section we give the proofs of the propositions, corollaries and lemmas stated earlier in the paper.

PROOF OF PROPOSITION 1.

- (i) A graph having a stepwise adjacency matrix is a nested split graph G . A nested split graph has a nested neighborhood structure. The neighborhood \mathcal{N}_j of an agent j is contained in the neighborhood \mathcal{N}_i of the next higher degree agent i with $|\mathcal{N}_i| = d_i > |\mathcal{N}_j| = d_j$ with $\mathcal{N}_j \subset \mathcal{N}_i$. For a symmetric adjacency matrix the vector of Bonacich centralities is given by $\mathbf{b}(G, \lambda) = \lambda \mathbf{A} \mathbf{b} + \mathbf{u}$, $\mathbf{u} = (1, \dots, 1)^T$. For agent i we get

$$b_i(G, \lambda) = \lambda \sum_{k=1}^n a_{ik} b_k(G, \lambda) + 1 = \lambda \sum_{k \in \mathcal{N}_i} b_k(G, \lambda) + 1, \quad (38)$$

and similarly for agent j

$$b_j(G, \lambda) = \lambda \sum_{k \in \mathcal{N}_j} b_k(G, \lambda) + 1. \quad (39)$$

Since $\mathcal{N}_j \subset \mathcal{N}_i$ and $d_j = |\mathcal{N}_j| < |\mathcal{N}_i| = d_i$ we get

$$\frac{b_i(G, \lambda)}{b_j(G, \lambda)} = \frac{\lambda \sum_{k \in \mathcal{N}_i} b_k(G, \lambda) + 1}{\lambda \sum_{k \in \mathcal{N}_j} b_k(G, \lambda) + 1} > 1. \quad (40)$$

The inequality follows from the fact that the Bonacich centrality is nonnegative and the numerator contains the sum over the same positive numbers as the denominator plus some additional values.

Conversely, in a nested split graph we must either have $\mathcal{N}_i \subset \mathcal{N}_j$ or $\mathcal{N}_j \subset \mathcal{N}_i$. Assuming that $b_i(G, \lambda) > b_j(G, \lambda)$ we can conclude from the above equation that $\mathcal{N}_j \subset \mathcal{N}_i$ and therefore $|\mathcal{N}_i| = d_i > |\mathcal{N}_j| = d_j$. If there are l distinct degrees in G then the ordering of degrees $d_1 > d_2 > \dots > d_l$ is equivalent to the ordering of the Bonacich centralities $b_1(G, \lambda) > b_2(G, \lambda) > \dots > b_l(G, \lambda)$.

- (ii) Consider the agents i, j and k in the nested split graph $G(t)$, such that $d_j \leq d_k$. Let G' be the graph obtained from $G(t)$ by adding the

link ij and G'' be the graph obtained from $G(t)$ by adding the link ik . We want to show that the Bonacich centrality of agent i in G'' is higher than in G' , that is, $b_i(G', \lambda) < b_i(G'', \lambda)$. For this purpose we count the number of walks emanating at agent i when connecting either to agent j or agent k . Since G is a nested split graph, we have

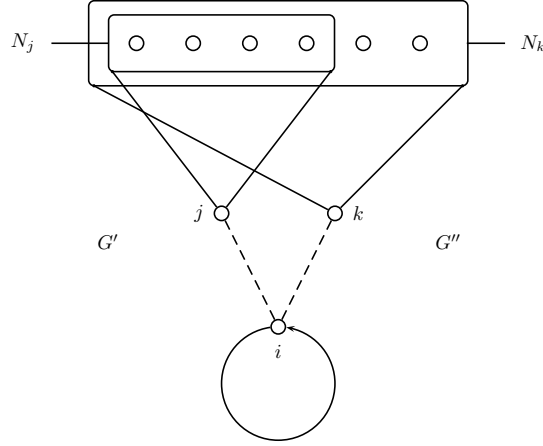


Figure 16: An illustration of the two networks G' and G'' , which differ in the links ij and ik . The neighborhood \mathcal{N}_j of agent j and the neighborhood \mathcal{N}_k of agent k are indicated by corresponding boxes. Note that the neighborhood of agent j is contained in the neighborhood of agent k . The loop at agent i indicates a walk starting at i and coming back to i before proceeding to either agent j or k .

that $\mathcal{N}_j \subset \mathcal{N}_k$. An illustration is given in Figure 16. We consider a walk W_l of length $l \geq 2$ starting at agent i in G' . We want to know how many such walks there are in G' and G'' , respectively. For this purpose we distinguish the following cases:

- (a) Assume that W_l does not contain the link ij nor the link ik . Then each such walk W_l in G' is also contained in G'' , since G' and G'' differ only in the links ij and ik .
- (b) Consider the graph G' and a walk W_l starting at agent i and proceeding to agent j . For each walk W_l in G' there exists a walk \tilde{W}_l in G'' being identical to W_l except of proceeding from i to j it proceeds from i to k and then to the neighbor of j that is visited after j in W_l . This is always possible since the neighbors of j are also neighbors of k .
- (c) Consider a walk W_l in G' that starts at i but first takes a detour returning to i before proceeding from i to j . Using the same argument as in (ii) it follows that for each such walk W_l in G' there exists a walk of the same length in G'' .
- (d) Consider a walk W_l in G' that starts at agent i and at some point

in its sequence of agents and links proceeds from agent j to agent i . For each such walk W_l in G' there exists a walk \tilde{W}_l in G'' that is identical to W_l except that it does not proceed from a neighbor of j to j and then to i it proceeds from a neighbor of j to k and then to i .

The above cases take into account all possible walks in G' and G'' of an arbitrary length l and show that in G'' there are at least as many walks of length l starting from agent i as there are in G' .

Now consider the walks of length two, W_2 , in G' starting at agent i and proceeding to agent j . Then there are $|\mathcal{N}_j|$ such walks in G' . However, there are $|\mathcal{N}_k| > |\mathcal{N}_j|$ such walks in G'' of length two that start at agent i .

The Bonacich centrality $b_i(G(t), \lambda)$ is computed by the number of all walks in $G(t)$ starting from i , where the walks of length l are weighted by their geometrically decaying factor λ^l . We have shown that for each l the number of walks in G'' is larger or equal than the number of walks in G' and for $l = 2$ it is strictly larger. Thus, the Bonacich centrality of agent i in G'' is higher than in G' . Note that all agents in a nested split graph are at most two links separated from each other (if there exists any walk between them). Thus, the agent with the highest degree is also the agent with the highest degree among the neighbors' neighbors. From this discussion we see that in a nested split graph $G(t)$ the local best response of an agent i are the agents with the highest degrees in i 's second-order neighborhood. \square

PROOF OF PROPOSITION 2. We give a proof by induction. Let $G(t)$ be a network generated by $((G(t))_{t=0}^\infty$. The induction basis is trivial. We start at $t = 0$ from an empty network $G(0) = \bar{K}_n$, which has a trivial stepwise adjacency matrix (see also the Definition 6). Since there are no link present in \bar{K}_n we can omit the removal of a link. At $t = 1$ we select an agent and connect her to another one. All isolated agents are best responses of the selected agent. This creates a path of length one whose adjacency matrix is stepwise. This is true because we can always find a simultaneous columns and rows permutation which makes the adjacency matrix stepwise. Thus $G(1)$ has a stepwise adjacency matrix.

Next we consider the induction step $G(t)$ to $G(t + 1)$. By the induction hypothesis, $G(t)$ is a nested split graph with a stepwise adjacency matrix. First, we consider the creation of a link ij . Now let agent j be a local best response of agent i , that is $j \in BR_i(G(t))$. Now, a link is created only if agent i is also a local best response of agent j , that is $i \in BR_j(G(t))$. Using Proposition 1, this means that agent i must be the agent with the highest degree in the second-order neighborhood of agent j . From the stepwise adjacency matrix $\mathbf{A}(G(t))$ of $G(t)$ (see Definition 6) we find that adding

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & \boxed{1} & 0 & 0 & \boxed{1} \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 17: Two possible positions for the creation of a link from agent 4, either to agent 7 or to agent 10, are indicated with boxes. Agent 7 has degree 3 while agent 10 has degree 1. Creating a link to an agent with higher degree results in higher equilibrium payoffs. Thus, the best response of agent 4 is agent 7 and not agent 10.

the link ij to the network $G(t)$ such that both agents are the agents with the highest degrees in their second-order neighborhoods results in a matrix $\mathbf{A}(G(t) + ij)$ that is stepwise. Therefore, the network $G(t) + ij$ is a nested split graph.

We give an example in Figure 17. Let the agents be numbered by the rows respectively columns of the adjacency matrix. We assume that agent 4 is selected to create a link. Two possible positions for the creation of a link from agent 4, either to agent 7 or to agent 10 are indicated with boxes. Since, in a stepwise matrix, the best response agent has the highest degree, agent 7 is a best response of agent 4 while agent 10 is not. We now can turn to the best response of agent 7. The agents not connected to agent 7 are indicated by zero entries in the seventh column of the adjacency matrix. There we find that agent 4 is also a best response of agent 7, since agent 4 is the agent with the highest degree not already connected to agent 7. Finally, we observe that creating the link 47 preserves the stepwise form of the adjacency matrix (see also Definition 6).³⁹

For the removal of a link a similar argument can be applied as in the preceding discussion. Disconnecting from the agent with the smallest degree decreases the Bonacich centrality and equilibrium payoffs the least. From the properties of the stepwise matrix $\mathbf{A}(G(t))$ it then follows that the matrix $\mathbf{A}(G(t) - ij)$ is stepwise.

Thus, in any step t in the network formation process $(G(t))_{t=0}^{\infty}$, $G(t)$ is a nested split graph with an associated stepwise adjacency matrix $\mathbf{A}(G(t))$. \square

³⁹The adjacency matrix is uniquely defined up to a permutation of its rows and columns. Applying such a permutation, we can always find an adjacency matrix which is stepwise.

PROOF OF COROLLARY 1. In Proposition 2 we have shown that $G(t)$ generated by $(G(t))_{t=0}^{\infty}$ is a nested split graph for all times t . In a nested split graph, any node in the connected component is directly connected to the node(s) with maximum degree. Thus, there exists a path of at most length two from any node to any other node in the connected component. It follows that $G(t)$ consists of a connected component and possible isolated nodes. \square

PROOF OF PROPOSITION 3. We will show that the network formation process $(G(t))_{t=0}^{\infty}$ introduced in Definition 3 induces a Markov chain on a finite state space Ω . Ω contains all unlabeled nested split graphs with n nodes. It can be shown that $|\Omega| = 2^{n-1}$ [Mahadev and Peled, 1995]. Therefore, the number of states is finite and the transition between states can be represented with a transition matrix \mathbf{P} . In the following we show that this Markov chain is irreducible and aperiodic. We then say that $(G(t))_{t=0}^{\infty}$ is ergodic.

First, we show that $(G(t))_{t=0}^{\infty}$ is a Markov chain. The network $G(t+1)$ is obtained from $G(t)$ by removing or adding a link to $G(t)$. Thus, the probability of obtaining $G(t+1)$ depends only on $G(t)$ and not on the previous networks $G(t')$ for $t' < t$, that is

$$\begin{aligned} \mathbb{P}(G(t+1) = G_j | G(0) = G_{i_0}, G(1) = G_{i_1}, \dots, G(t) = G_{i_t}) = \\ \mathbb{P}(G(t+1) = G_j | G(t) = G_{i_t}). \end{aligned} \quad (41)$$

The number of possible networks $G(t)$ is finite for any time t and the transition probabilities from a network $G(t)$ to $G(t+1)$ do not depend on t but only on α and the current number of agents in the independent sets and dominating sets, respectively. Therefore, $(G(t))_{t=0}^{\infty}$ is a finite state, discrete time, homogeneous Markov chain. Moreover, the transition matrix \mathbf{P} is defined by $(\mathbf{P})_{ij} = \mathbb{P}(G(t+1) = G_j | G(t) = G_i)$ for any $G_i, G_j \in \Omega$.

Next, we show that the Markov chain is irreducible. Consider two networks $G, G' \in \Omega$. $(G(t))_{t=0}^{\infty}$ is irreducible if there exists a positive probability to pass from any G to any other G' in Ω . We say that G' is accessible from G . For any G there exists a positive probability that in all consecutive steps in the Markov chain links are removed and no links are created until the empty network $\bar{K}_n \in \Omega$ is reached. Then there exists a positive probability that from \bar{K}_n only those links are created that generate exactly the network G' . Therefore, there exists a positive probability to pass from any network G to any other network G' with positive probability. Similarly, one can show that G is accessible from G' . States G and G' are accessible from one-another. We say that they communicate and Ω is a communicating class.

Moreover, the Markov chain is aperiodic. Observe that with positive probability the empty network \bar{K}_n can stay empty in the next time step. This happens when an agent in \bar{K}_n is selected for removing a link with probability $1 - \alpha$. Since this agent has no links, nothing happens. Thus, the state

\bar{K}_n is aperiodic. The existence of an aperiodic state in the communicating class Ω implies that the Markov chain induced by $(G(t))_{t=0}^\infty$ is aperiodic.

Since we have shown that $(G(t))_{t=0}^\infty$ induces a finite Markov chain that is irreducible and aperiodic, we say that the Markov chain is ergodic. Further, this means that there exists a unique stationary distribution μ satisfying $\mu\mathbf{P} = \mu$ [see e.g. [Seneta, 1973, 2006](#)]. \square

PROOF OF PROPOSITION 4. We consider the network formation process $(G(t))_{t=0}^\infty$ on Ω introduced in Definition 3. At every step $t = 0, 1, 2, \dots$ a link is created with probability α and a link is removed with probability $1 - \alpha$. Further, we consider the complementary network formation process $(G'(t))_{t=0}^\infty$ on Ω where in every period t a link is created with probability $\alpha' = 1 - \alpha$ and a link is removed with probability $1 - \alpha' = \alpha$.⁴⁰ This means that a link is removed in $(G'(t))_{t=0}^\infty$ whenever a link is created in $(G(t))_{t=0}^\infty$ and a link is created whenever a link is removed in $(G(t))_{t=0}^\infty$.

As an example, consider the network G represented by the adjacency matrix \mathbf{A} in Figure 17. The complement \bar{G} has an adjacency matrix $\bar{\mathbf{A}}$ obtained from \mathbf{A} by replacing each one element in \mathbf{A} by zero and each zero element by one, except for the elements on the diagonal. Let H be the network obtained from G by adding the link 47 (setting $a_{47} = a_{74} = 1$ in \mathbf{A}). The probability of this link being created and thus the probability of reaching H after the process was in G is $3\alpha/n$, either by selecting one of the two nodes with degrees three or the node with degree five to create a link. Observe that this is identical to the probability of reaching the network \bar{H} from \bar{G} if either the two nodes with degrees seven or the node with degree four in \bar{G} are selected to remove a link (with probability $\alpha' = 1 - \alpha$).

In general we can say that, for any $G_1, G_2 \in \Omega$ we have that

$$\mathbb{P}(G(t+1) = G_2 | G(t) = G_1) = \mathbb{P}(G'(t+1) = \bar{G}_2 | G'(t) = \bar{G}_1). \quad (42)$$

Next consider the stationary distribution μ of $(G(t))_{t=0}^\infty$ and the corresponding transition matrix \mathbf{P} . Similarly, consider the stationary distribution μ' of $(G'(t))_{t=0}^\infty$ and the corresponding transition matrix \mathbf{P}' . Further, consider an ordering of states G_1, G_2, \dots in Ω and the transition matrix \mathbf{P} with elements $(\mathbf{P})_{ij}$ giving the probability of observing G_j after the Markov chain $(G(t))_{t=0}^\infty$ was in G_i . Similarly, consider an ordering of states $\bar{G}_1, \bar{G}_2, \dots$ in

⁴⁰Two nodes of $G'(t)$ are adjacent if and only if they are not adjacent in $G(t)$. Note that the complement of a nested split graph is a nested split graph as well [[Mahadev and Peled, 1995](#)]. In particular, the networks $G'(t)$ are nested split graphs in which the number of nodes in the dominating sets corresponds to the number of nodes in the independent sets in $G(t)$ and the number of nodes in the independent sets in $G'(t)$ corresponds to the number of nodes in the dominating sets in $G(t)$. Thus, $(G'(t))_{t=0}^\infty$ has the same state space Ω as $(G(t))_{t=0}^\infty$, namely the space consisting all unlabeled nested split graphs on n nodes.

Ω and the transition matrix \mathbf{P}' with elements $(\mathbf{P}')_{ij}$ giving the probability of observing \bar{G}_j after the Markov chain $(G'(t))_{t=0}^\infty$ was in \bar{G}_i . Equation (42) implies that $\mathbf{P} = \mathbf{P}'$. Moreover, for the stationary distributions it must hold that $\mu\mathbf{P} = \mu$ and $\mu'\mathbf{P}' = \mu'$. Since \mathbf{P} is irreducible and aperiodic, \mathbf{P} has a unique positive eigenvector and therefore $\mu' = \mu$. It follows that for any network $G \in \Omega$ with probability μ_G we can take the complement $\bar{G} = G'$ and assign it the probability μ_G to get the corresponding probability in μ' , i.e. $\mu_G = \mu'_{G'}$. \square

PROOF OF PROPOSITION 5. Before we proceed with the proof of Proposition 5, we state two useful lemmas.

Lemma 1. *Consider the ergodic Markov chain $(G(t))_{t=0}^\infty$ with the parameter $0 < \alpha \leq 1/2$ and state space Ω consisting of all nested split graphs. Let X denote the set of states in Ω in which there is exactly one node with degree $d + 1$ and Y the set of states where there is no node with degree $d + 1$. Denote by μ_X the probability of the states in X in the stationary distribution μ of $(G(t))_{t=0}^\infty$ and by μ_Y the probability of states in Y . If the number of nodes with degree N_d in Y is $\Theta(n)$ such that $\lim_{n \rightarrow \infty} N_d/n > 0$ then $\lim_{n \rightarrow \infty} \mu_Y = 0$.⁴¹*

PROOF OF LEMMA 1. Let $N(X, Y, y)$ be the expected number of times states in X occur before the process reaches Y (not counting the process as having immediately reached Y if $y \in Y$) when the process starts in y . Then the following relation holds (see Theorem 6.2.3 in [Kemeny and Snell \[1960\]](#) and also [Ellison \[2000\]](#))

$$\frac{\mu_X}{\mu_Y} = N(X, Y, y). \quad (43)$$

Let p_{YX} denote a lower bound on the probability that a state in X occurs after the process is in a state in Y and, conversely, let p_{XY} denote the probability that a state in Y occurs after the process is in a state in X . This probability is the same for all states in X , since from the properties of the Markov chain $(G(t))_{t=0}^\infty$, it follows that $p_{XY} = 2(1 - \alpha)/n$, because there exist two possibilities to remove the link of the node with degree $d + 1$ and the probability to select a node for link removal is $(1 - \alpha)/n$. Observe that this probability vanishes for large n , $\lim_{n \rightarrow \infty} p_{XY} = 0$. Moreover, we

⁴¹By $f = \Theta(g)$ we mean that

$$0 < \liminf_{n \rightarrow \infty} \left| \frac{f(n)}{g(n)} \right| \leq \limsup_{n \rightarrow \infty} \left| \frac{f(n)}{g(n)} \right| < \infty.$$

In particular, $f = \Theta(1)$ implies that $0 < \lim_{n \rightarrow \infty} f(n) < \infty$.

have that

$$\begin{aligned}
N(X, Y, y) &\geq p_{YX}p_{XY} \\
&\quad + 2p_{YX}(1 - p_{XY})p_{XY} \\
&\quad + 3p_{YX}(1 - p_{XY})^2p_{XY} \\
&\quad + \dots \\
&= p_{YX}p_{XY} \sum_{i=1}^{\infty} i(1 - p_{XY})^{i-1} \\
&= \frac{p_{YX}}{p_{XY}}.
\end{aligned}$$

The right hand side of the above inequality takes into account the fact that states in X can be reached once, twice, etc., before a state in Y is reached and assigns the corresponding probabilities to compute the expected value.

By assuming that there exists a number N_d of nodes with degree d which is $\Theta(n)$, we have that $p_{YX} \geq \alpha N_d/n$ and $\lim_{n \rightarrow \infty} p_{YX} > 0$. It then follows that

$$\frac{\mu_X}{\mu_Y} = N(X, Y, y) \geq \frac{p_{YX}}{p_{XY}} = \frac{\alpha}{2(1 - \alpha)} N_d \xrightarrow[n \rightarrow \infty]{} \infty. \quad (44)$$

Since μ_X is a probability with $\mu_X \leq 1$, Equation (44) implies that $\lim_{n \rightarrow \infty} \mu_Y = 0$. \square

Lemma 2. For $0 < \alpha \leq 1/2$ the asymptotic expected proportion of isolated nodes in the limit of large n is given by

$$n_0 = \frac{1 - 2\alpha}{1 - \alpha}. \quad (45)$$

PROOF OF LEMMA 2. We consider the expected change in the number of links $m(t)$ in $G(t)$ from t to $t + 1$.⁴² The number of links increases by one if any node which does not have the maximum degree $n - 1$ is selected for creating a link. This happens with probability $\alpha (n - N_{n-1}(t)) / n$. The number of links decreases whenever a node with degree higher than zero is selected for removing a link. This happens with probability $(1 - \alpha) (n - N_0(t)) / n$. Putting the above contributions together we can write for the expected change in the total number of links from t to $t + 1$

$$\mathbb{E}(m(t + 1) | N(t)) - m(t) = \frac{\alpha}{n} (n - N_{n-1}(t)) - \frac{1 - \alpha}{n} (n - N_0(t)). \quad (46)$$

Taking expectations on both sides of the above equation and denoting by

⁴²We have that $2m(t) = \sum_{d=0}^{n-1} N_d(t)d$.

$n_d(t) = \mathbb{E}(N_d(t)/n)$ we obtain

$$\mathbb{E}(m(t+1)) - \mathbb{E}(m(t)) = \alpha(1 - n_{n-1}(t)) - (1 - \alpha)(1 - n_0(t)). \quad (47)$$

Let ρ denote the initial distribution of states, with $\rho_i = 1$ if $G_i = \bar{K}_n$ and zero otherwise. Further, let m be the column vector whose j -th coordinate, m_j , is the value of m at state $G_j \in \Omega$. Let $G_i = \bar{K}_n$ then we can write

$$\begin{aligned} \mathbb{E}(m(t)) &= \mathbb{E}(m(t)|G(0) = G_i) \\ &= \sum_{G_j \in \Omega} m_j \mathbb{P}(G(t) = G_j | G(0) = G_i) \\ &= \sum_{G_j \in \Omega} (\mathbf{P}^t)_{ij} (m)_j = (\mathbf{P}^t m)_i = \rho \mathbf{P}^t m. \end{aligned}$$

For large times t the expectation is computed over the invariant distribution μ . In particular, $\lim_{t \rightarrow \infty} \rho \mathbf{P}^t = \mu$ and therefore $\lim_{t \rightarrow \infty} \mathbb{E}(m(t)) = \lim_{t \rightarrow \infty} \rho \mathbf{P}^t m = \mu m$. Similarly, we have that $\lim_{t \rightarrow \infty} \mathbb{E}(m(t+1)) = \lim_{t \rightarrow \infty} \rho \mathbf{P}^{t+1} m = \mu \mathbf{P} m = \mu m$. Therefore, $\lim_{t \rightarrow \infty} \mathbb{E}(m(t+1)) = \lim_{t \rightarrow \infty} \mathbb{E}(m(t))$. Thus, we can set the left hand side of Equation (46) to zero, in the limit of large t , and obtain a relationship between the asymptotic expected proportion of nodes of degree zero and one, respectively,

$$1 - 2\alpha = (1 - \alpha)n_0 - \alpha n_{n-1}, \quad (48)$$

where we have denoted by $n_d = \lim_{t \rightarrow \infty} n_d(t)$. Next, we consider the chain $(G'(t))_{t=0}^{\infty}$ which is constructed from $(G(t))_{t=0}^{\infty}$ by taking the complement of each network $G(t)$ in every period t (see also the proof of Proposition 4). In the following, denote the asymptotic expected number of links $\lim_{t \rightarrow \infty} \mathbb{E}(m(t))$ of $(G(t))_{t=0}^{\infty}$ by m and of $(G'(t))_{t=0}^{\infty}$ by m' . By construction, we must have that $m = n(n-1)/2 - m'$. From Proposition 4 we know that the Markov chain $(G'(t))_{t=0}^{\infty}$ has the same stationary distribution μ' as a process from Definition 3 for a link creation probability of $\alpha' = 1 - \alpha$. For $\alpha = 1/2$ the two processes are identical and we must have that also their expected number of links are the same. This implies that for $\alpha = 1/2$, $m = m' = n(n-1)/4$. The only nested split graph with this number of links, for which the complement has the same number of links as the original graph, is the one in which each independent set is of size one and also each dominating set has size one (except possibly for the set corresponding to the $(\lfloor \frac{k}{2} \rfloor + 1)$ -th partition). Thus, for $\alpha = 1/2$ it must hold that $n_0 = n_{n-1} = 1/n$.

Moreover, we know that for $\alpha < 1/2$ the expected number of maximally connected nodes (with degree $n-1$) is at most as large as the expected number for $\alpha = 1/2$, since the probability of links being created strictly decreases while the probability of links being removed increases for values

of α below $1/2$ (and the probability of a maximally connected node losing a link strictly increases). Thus $n_{n-1} \leq 1/n$ for $\alpha \leq 1/2$, and for large n we can write Equation (48) as follows

$$1 - 2\alpha = (1 - \alpha)n_0, \quad (49)$$

or equivalently

$$n_0 = \frac{1 - 2\alpha}{1 - \alpha}. \quad (50)$$

For $\alpha = 0$ no links are created and all nodes are isolated, that is $n_0 = 1$, while for $\alpha = 1/2$ the asymptotic expected number of isolated nodes vanishes in the limit of large n . \square

With these two lemmas in hand, let us now prove Proposition 5. Note that $G(t)$ is completely determined by $N(t)$ and vice versa. Thus it follows that $\{N(t)\}_{t=0}^{\infty}$ is a Markov chain. Denote by $n_d(t) = \mathbb{E}(N_d(t)/n)$ the expected proportion of nodes with degree d at time t and let us denote by $n_d = \lim_{n \rightarrow \infty} n_d(t)$; n_d is determined by the invariant distribution μ in the limit of large times t . Lemma 2 shows that Equation (7) holds for $d = 0$. In the following we show by induction that, given that Equation (7) holds for n_{d-1} and n_d , as n becomes large, also n_{d+1} satisfies Equation (7) for all $0 \leq d < d^*$, in the limit of large n . For this purpose we consider (a) the expected number of isolated nodes $\mathbb{E}(N_0(t+1)|N(t))$ and (b) the expected number of nodes with degree $d = 1, \dots, d^*$, $\mathbb{E}(N_d(t+1)|N(t))$ at time $t+1$, conditional on the current degree distribution $N(t)$.

(a) Consider a particular network $G(t)$ in period t generated by $(G(t))_{t=0}^{\infty}$ and its associated degree distribution $N(t)$. Figure 18 shows an illustration of the corresponding stepwise matrix. In the following we compute the expected change of the number $N_0(t)$ of isolated nodes in $G(t)$.

The expected change of $N_0(t)$ due to the creation of a link has the following contributions. An agent with the highest degree k in $N_k(t)$ can create a link to an isolated agent and thus decreases the number of isolated agents by one. The expected change from this link is $-\alpha N_k(t)/n$. On the other hand, if an isolated agent creates a link then the expected change in the number of isolated agents is $-\alpha N_0(t)/n$.

Moreover, the removal of links can affect $N_0(t)$ if there is only one agent with maximal degree, i.e. $N_k(t) = 1$. In this case, if the agent with the highest degree removes a link, then an additional isolated agent is created yielding an expected increase in $N_0(t)$ of $(1 - \alpha)N_k(t)/n$. Next, if an agent with degree one in $N_1(t)$ removes a link, then the number of isolated agents increases. Note that in a nested split graph $N_1(t) > 0$ implies that $N_k(t) = 1$ and vice versa. This gives an expected change of $N_0(t)$ given by $(1 - \alpha)N_1(t)/n$.

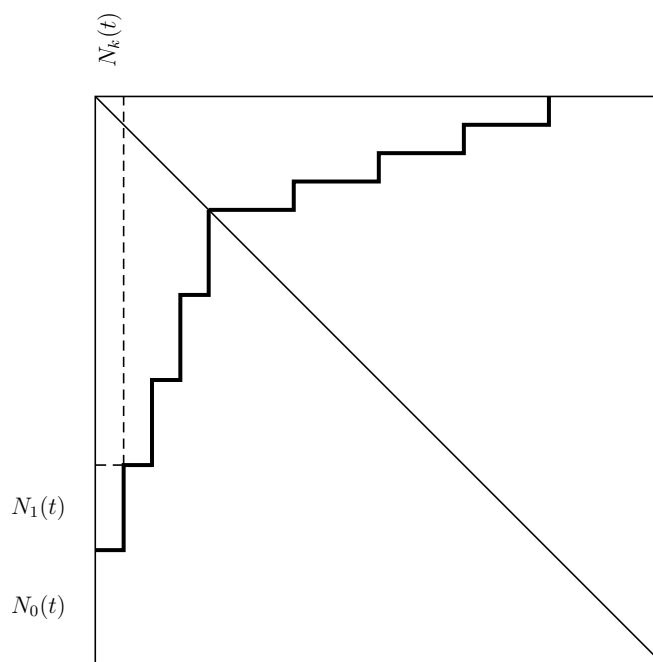


Figure 18: Representation of the stepwise matrix \mathbf{A} of a nested split graph G and some selected degree partitions. The stepfunction separating the zero entries in the matrix from the one entries is shown with a thick line.

Putting the above contributions together, the expected change in the number of isolated nodes at time $t + 1$, conditional on $N(t)$, is given by the following expression⁴³

$$\begin{aligned} \mathbb{E}(N_0(t+1)|N(t)) - N_0(t) = \\ -\frac{\alpha}{n}(N_0(t) + N_k(t)) + \frac{1-\alpha}{n}(N_1(t) + 1)\delta_{N_k(t),1}. \end{aligned} \quad (51)$$

We can take expectations on both sides of Equation (51). For large times t the expectation is computed on the basis of the invariant distribution μ and similarly to the proof of Lemma 2, after taking expectations, we can set the left hand side of Equation (51) to zero for large times t . Note that from Lemma 2 we know that the asymptotic expected proportion n_0 of isolated nodes is $\Theta(1)$, for n large. Thus we can apply the result of Lemma 1 which tells us that the networks in which there does not exist a node with degree one have vanishing probability in μ for large n . Since the existence of a node with degree one implies that $N_k(t) = 1$, in the limit of large n we can set $\delta_{N_k(t),1} = 1$. We then obtain from Equation (51)

$$n_1 = \frac{\alpha}{1-\alpha}n_0. \quad (52)$$

This shows that also n_1 satisfies Equation (7). Together with Lemma 2 this proves the induction basis.

- (b) We give a proof by induction on the number $N_d(t)$ of nodes with degree $0 < d < d^*$ in a network $G(t)$ in the support of the stationary distribution μ . In the following, we compute the expected change in $N_d(t)$ due to the creation or the removal of a link. An illustration can be found in Figure 19.

Let us investigate the creation of a link. With probability α/n a link is created from an agent in $N_{k-d}(t)$ to an agent in $N_d(t)$. This yields a contribution to the expected change of $N_d(t)$ of $-\alpha N_{k-d}(t)/n$. If a link is created from an agent in $N_{k-d+1}(t)$ to an agent in $N_d(t)$ then the expected change is α/n , if $N_{k-d+1}(t)$ contains only a single agent. Similarly, if a link is created from an agent in $N_{d-1}(t)$ to an agent in $N_d(t)$ then the expected change of $N_d(t)$ is $\alpha N_{d-1}(t)/n$, if $N_{k-d+1}(t) = 1$. Moreover, if an agent in $N_d(t)$ is selected for link creation then we get an expected decrease of $-\alpha N_d(t)/n$.

Now we consider the removal of a link. If a link is removed from the agent in $N_{k-d+1}(t)$ to an agent in $N_d(t)$ then the expected change of $N_d(t)$ is $-(1-\alpha)N_{k-d+1}(t)/n$. If a link is removed from an agent in $N_{k-d}(t)$ to an agent in $N_{d+1}(t)$ then the expected increase of $N_d(t)$ is

⁴³ $\delta_{i,j}$ denotes the usual Kronecker delta which is 1 if $i = j$ and 0 otherwise.

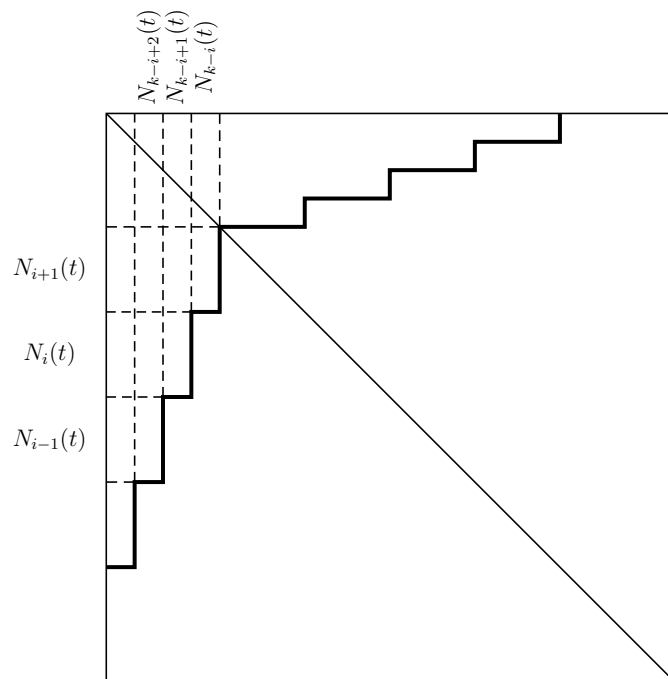


Figure 19: Representation of the stepwise matrix \mathbf{A} of a nested split graph G . The stepfunction separating the zero entries in the matrix from the one entries is shown with a thick line.

$(1-\alpha)/n$, if $N_{k-d}(t) = 1$. Moreover, if an agent in $N_{d+1}(t)$ is selected for removing a link, then we get an expected increase of $(1-\alpha)N_{i+d}(t)/n$, if $N_{k-d}(t) = 1$. Finally, if an agent in $N_d(t)$ is selected for removing a link, then we get an expected change of $-(1-\alpha)N_d(t)/n$. Putting the above contributions together, the expected change in $N_d(t)$ is given by

$$\begin{aligned} \mathbb{E}(N_d(t+1)|N(t)) - N_d(t) = & \\ & \frac{\alpha}{n} (-N_d(t) + (N_{d-1}(t) + 1) \delta_{N_{k-d+1}(t),1} - N_{k-d}(t)) \\ & + \frac{1-\alpha}{n} (-N_d(t) + (N_{d+1}(t) + 1) \delta_{N_{k-d}(t),1} - N_{k-d+1}(t)). \end{aligned} \quad (53)$$

We can take expectations on both sides of Equation (53) and similarly to part (a) of this proof we can set the left-hand-side of Equation (53) as t becomes large. For large times t the above expectation is computed on the basis of the invariant distribution μ . By the induction assumption, the asymptotic expected proportion n_{d-1} of nodes with degree $d-1$ is $\Theta(1)$ in the limit of large n (as follows from Equation (7)). Thus we can apply Lemma 1 and neglect the networks in which there does not exist a node with degree d since they have vanishing probability in μ for large n . Similarly, we know from the induction assumption that the asymptotic proportion n_d of nodes with degree d is $\Theta(1)$ and, by virtue of Lemma 1, we know that the networks in which there does not exist a node with degree $d+1$ have vanishing probability in μ for large n . Thus, in the limit of large n we can set $\delta_{N_{k-d+1}(t),1} = \delta_{N_{k-d}(t),1} = 1$, since the existence of nodes with degrees d and $d+1$ implies that $N_{k-d+1}(t) = N_{k-d}(t) = 1$ in the limit of large t and n . Therefore, we get from Equation (53) the following relationship

$$n_{d+1} = \frac{1}{1-\alpha} n_d - \frac{\alpha}{1-\alpha} n_{d-1}. \quad (54)$$

Inserting the expressions for n_{d-1} and n_d from Equation (7) into Equation (54) yields

$$\begin{aligned} n_{d+1} &= \frac{1}{1-\alpha} \frac{1-2\alpha}{1-\alpha} \left(\frac{\alpha}{1-\alpha} \right)^d - \frac{\alpha}{1-\alpha} \frac{1-2\alpha}{1-\alpha} \left(\frac{\alpha}{1-\alpha} \right)^{d-1} \\ &= \frac{1-2\alpha}{1-\alpha} \left(\frac{\alpha}{1-\alpha} \right)^{d+1} \end{aligned}$$

Thus, Equation (7) also holds for n_{d+1} . This proves the induction step.

Finally, we have that the degree distribution must be normalized to one, i.e. $\sum_{d=0}^{n-1} n_d = 1$. We know that the number of agents in the dominating

sets with degrees larger than d^* is d^* (since each set contains only one node and there are d^* such sets).⁴⁴ Adding this to the number of agents in the independent sets with degree $d = 0, \dots, d^*$ yields

$$n \sum_{d=0}^{d^*} n_d + d^* = n. \quad (55)$$

Further, inserting Equation (7) we can derive the number d^* of independent sets as a function of n and α

$$d^*(n, \alpha) = \frac{\ln \left(\frac{2(1-\alpha)}{(1-2\alpha)n} \right)}{\ln \left(\frac{\alpha}{1-\alpha} \right)}. \quad (56)$$

d^* is a monotonic decreasing function of n for a fixed value of α . Conversely, for a fixed value of n we get the limits $\lim_{\alpha \rightarrow 0} d^* = 0$ and $\lim_{\alpha \rightarrow 1/2} d^* = n/2$. This completes the proof. \square

PROOF OF COROLLARY 2. The results follows directly from the functional form of $d(n, \alpha)$ in Proposition 5. \square

PROOF OF PROPOSITION 6. Let us define the following random variable

$$Y_s = \mathbb{E}(N_d(t) | N(s)). \quad (57)$$

Since $\{N(t)\}_{t=0}^{\infty}$ is a Markov chain, the sequence $\{Y_s\}_{s=0}^t$ is a Martingale with respect to $\{N(t)\}_{t=0}^{\infty}$.⁴⁵ Moreover, the change in the number of nodes with degree d per period t is bounded by two, i.e. $|N_d(t) - N_d(t-1)| \leq 2$, since at most one link is added or removed in every period t and this can change the degrees of at most two nodes. Therefore, we can apply Hoeffding's inequality [Hoeffding, 1963], which states that for any $0 < s \leq t$ with $|Y_s - Y_{s-1}| \leq c$ and any $\epsilon > 0$

$$\mathbb{P}(|Y_t - Y_0| > \epsilon) \leq 2e^{-\frac{\epsilon^2}{2tc^2}}. \quad (58)$$

With $Y_t = \mathbb{E}(N_d(t) | N(t)) = N_d(t)$ and $Y_0 = \mathbb{E}(N_d(t) | N(0)) = \mathbb{E}(N_d(t))$ it

⁴⁴Note that since those networks in which there does not exist a node with degree $0 \leq d \leq d^*$ in the corresponding independent set can be neglected, the structure of nested split graphs implies that all dominating sets have size one.

⁴⁵We have that $\mathbb{E}(Y_s | N(s-1)) = \mathbb{E}(\mathbb{E}(N_d(t) | N(s)) | N(s-1)) = \mathbb{E}(N_d(t) | N(s-1)) = Y_{s-1}$. Further, one can show that the first and second moments of $\{Y_s\}_{s=0}^t$ are bounded. Thus, $\{Y_s\}_{s=0}^t$ is a Martingale with respect to $\{N(t)\}_{t=0}^{\infty}$ [see also Grimmett and Stirzaker, 2001].

follows from Equation (58) that

$$\mathbb{P} \left(\left| \frac{N_d(t)}{n} - \mathbb{E} \left(\frac{N_d(t)}{n} \right) \right| \geq \epsilon \right) = \mathbb{P} (|N_d(t) - \mathbb{E}(N_d(t))| \geq n\epsilon) \leq 2e^{-\frac{\epsilon^2 n^2}{8t}}. \quad (59)$$

This implies that the empirical proportion $N_d(t)/n$ of nodes with degree d converges in probability to its expected value $\mathbb{E}(N_d(t)/n)$, as n becomes large.

PROOF OF COROLLARY 3. See Theorem 1.2.4 in [Mahadev and Peled \[1995\]](#). \square