Dynamics of Companies

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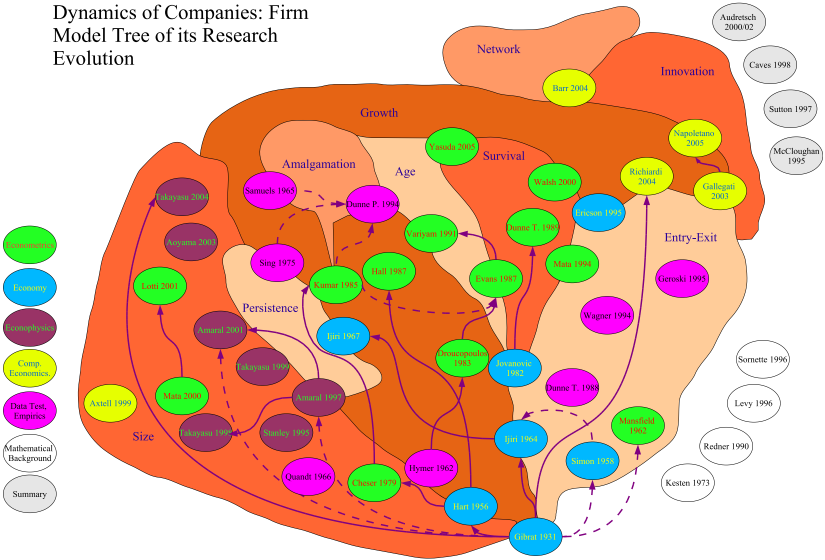
Thanks to ...

L. Amaral, H. Takayasu, S. Jain, S. Battiston, B. Drossel

Topics of Interest

- different perspectives:
  - economic: theory of the firm, ...
  - management: strategies for entrepreneurship, ...
  - production: supply chains, ...
- physics perspective: *collective* effects
  - ensembles of companies: $i = 1, ..., N$
  - simple characterization: company "size" $x_i(t)$
    income, output, employees, ...
  - aggregated variables
- our focus:
  - growth of companies ⇒ size *distribution*
  - interaction of companies ⇒ network structure
  - structure of companies ⇒ hierarchies
  - decisions in companies ⇒ opinion formation


Dynamics of Companies: Firm Model Tree of its Research Evolution
Network
Innovation
Growth
Amalgamation
Age
Survival
Entry-Exit
Persistence
Size
Econometrics
Economy
Econophysics
Comp. Economics.
Data Test, Empirics
Mathematical Background
Summary
Audretsch 2000/02
Caves 1998
Sutton 1997
McCloughan 1995
Sornette 1996
Levy 1996
Redner 1990
Kesten 1973
Barr 2004
Napoletano 2005
Richiardi 2004
Gallegati 2003
Yasuda 2005
Walsh 2000
Ericson 1995
Dunne T. 1989
Mata 1994
Geroski 1995
Wagner 1994
Samuels 1965
Dunne P. 1994
Variyam 1991
Evans 1987
Takayasu 2004
Aoyama 2003
Sing 1975
Kumar 1985
Hall 1987
Lotti 2001
Amaral 2001
Ijiri 1967
Takayasu 1999
Droucopoulos 1983
Jovanovic 1982
Dunne T. 1988
Axtell 1999
Mata 2000
Amaral 1997
Takayasu 1998
Stanley 1995
Mansfield 1962
Ijiri 1964
Simon 1958
Quandt 1966
Cheser 1979
Hymer 1962
Hart 1956
Gibrat 1931

Company Growth

- set of companies: $i = 1, \ldots, N$
  - $x_i(t)$: company "size", growth rate: $dx_i/dt = \mathcal{F}_i(?)$
- $\mathcal{F}_i(t)$ with $\langle \mathcal{F}_i(t) \rangle = 0$, $\langle \mathcal{F}_i(t)\mathcal{F}_i(t') \rangle = S\delta_{ij}\delta(t - t')$

$$x_i(t + \Delta t) = x_i(t) + \sqrt{S\Delta t}\,\xi_i$$

  - growth as random walk (Bachelier, 1900)
- $\mathcal{F}_i = f(x_i) + \ldots \Rightarrow$
  - independent growth, proportional to size (Gibrat, 1931)
- $\mathcal{F}_i = f(x_j, x_k) + \ldots$
  - growth through innovation networks

Gibrat's Model

- $\dot{x}_i = \mathcal{F}_i = f(x_i) + \ldots = b_i\, x_i$
  - "Law of proportionate growth" (Gibrat '30, '31; Sutton '97)
  - no interactions between firms

$$x_i(t + \Delta t) = x_i(t)\Big[1 + b_i(t)\Big]$$

- Assumptions:
  - $b_i(t)$: independent of i, no temporal correlations (random noise)
  - $t \gg \Delta t$:
    $x(t) = x(0)\big(1 + b(1)\big)\big(1 + b(2)\big)\cdots\big(1 + b(t)\big)$
  - *growth "rates":* $R(t) = x(t+1)/x(t)$, $t \gg \Delta t$, $\ln(1+b) \approx b$

$$\ln R(t) = \sum_{n=1}^{t} b(n)$$

$\Rightarrow$ random walk for $\ln R(t)$ $\Rightarrow$ log-normal distribution for $x_i(t)$

Log-normal vs Power Law Distribution

$$x_{t+1} = \lambda_t \, x_t \quad \text{with } \lambda = b + 1$$

$$P(x_t) = \frac{1}{\sqrt{2\pi D t}} \frac{1}{x_t} \exp\left[-\frac{1}{2Dt} (\log x_t - vt)^2 \right]$$

$$v = \langle \log \lambda \rangle \; ; \quad D = \langle (\log \lambda)^2 \rangle - \langle \log \lambda \rangle^2$$

rewriting:

$$P(x_t) = \frac{1}{\sqrt{2\pi D t}} \frac{1}{x_t^{1+\mu(x_t)}} e^{\mu(x_t) vt} \; ; \quad \mu(x_t) = \frac{1}{2Dt} \log \frac{x_t}{e^{vt}}$$

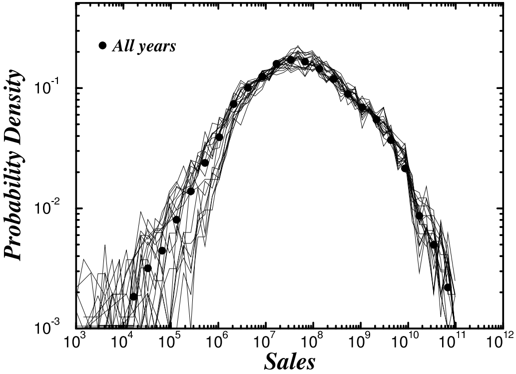
$\mu(x_t)$: slowly varying function of x_t

- $x_t \ll e^{(v+2D)t}$ yields $\mu(x_t) \ll 1$
  $\Rightarrow$ log-normal and $1/x_t$ undistuingishable
- however in the tail $x_t \gg e^{(v+2D)t} \Rightarrow \mu(x_t) \to \infty$ (!!)

Empirical Evidence?

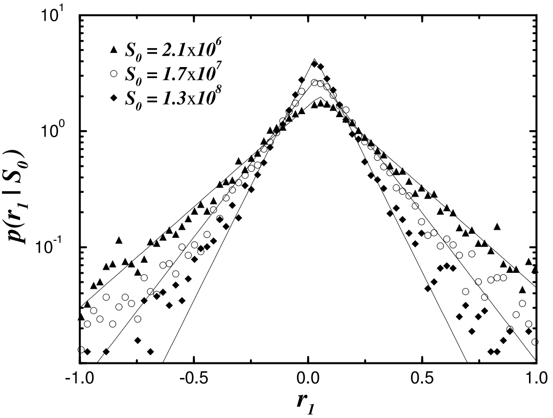
- Empirical distribution of company sizes (1974-1993) (Amaral et al, 1997) ⇒ log-normal distribution

Fig. 2(a)

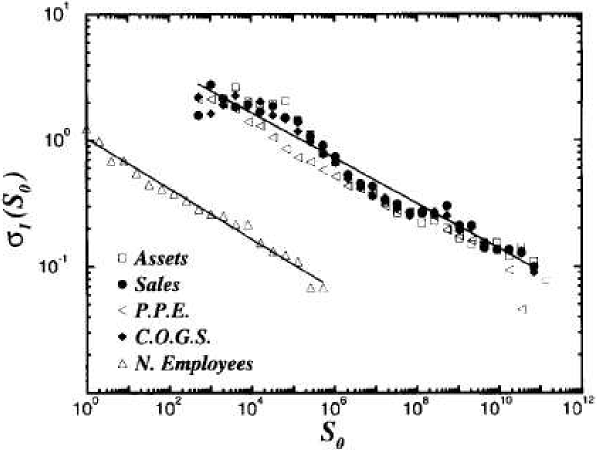


- Empirical distribution of growth rates (Amaral et al, 1997)
  ⇒ depend on size, tent-shape exponential distribution

Fig. 3



- Empirical distribution of standard deviation of growth rates (Amaral et al, 1997) ⇒ depend on size, power-law distribution



Stylized facts:

- log-normal distribution of company sizes

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma\, x} \exp\left[\frac{(-\ln x - \mu)^2}{2\sigma^2}\right]$$

- exponential growth ratio distribution

$$P(r_1|x_0) = \frac{1}{\sqrt{2}\,\sigma_1(x_0)} \exp - \left[\frac{\sqrt{2}\,|r_1 - \bar{r}_1(x_0)|}{\sigma_1(x_0)}\right]$$

- power-law distribution of the standard deviations

$$\sigma_1(x_0) \sim x_0^{-\beta}\ ; \quad \beta < 0.5$$

Explanations

- correlations in the growth rates
  company is attracted to an "optimal size"

$$\frac{x_{t+\Delta t}}{x_t} = \begin{cases} k e^{\varepsilon_t}, & x_t < x^* \\ \frac{1}{k} e^{\varepsilon_t}, & x_t > x^*, \end{cases}$$

- growth depends on properties of management hierarchies
  n levels, z mean branching ratio, decisions on higher level are followed with prob π

$$\beta = \begin{cases} -\ln(\pi)/\ln(z) & \text{if} \quad \pi > z^{-1/2} \\ 1/2 & \text{if} \quad \pi < z^{-1/2} \end{cases}$$

  - β decreases in time $\Leftrightarrow$ companies better coordinated

Micromodel of company growth

- Amaral et al (2001)
  - firm consist of several subunits (divisions): $\xi(t)$
  - firm size: $S(t) \equiv \sum_i \xi_i(t)$
    growth from independent growth of subunits
  - *entry:* $t = 0$: firm created with single unit of size $\xi_1(t = 0)$
  - *exit:* $S < S_{min}$: firm not economically viable

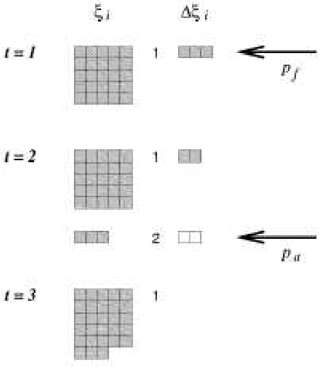
The evolution of a division

- random multiplicative process:

$$\Delta\xi_i(t) \equiv \xi_i(t)\nu_i(t)$$

- $\Delta\xi_i(t) < S_{min}$: division evolves by changing its size $\xi_i(t+1) = \xi_i(t) + \Delta\xi_i(t)$
  - if $\xi_i < S_{min}$, then with probability p_a division i is absorbed by division 1
- $\Delta\xi_i(t) > S_{min}$
  - with probability p_f division i does not change and a new division j is created with size $\xi_j(t+1) = \Delta\xi_i(t)$
  - with probability $(1-p_f)$ division i evolves $\xi_i(t+1) = \xi_i(t) + \Delta\xi_i(t)$,

Schematic representation of Amaral et al.'s model



Further Improvements of Gibrat

- *economic idea:* simple entry dynamics (Simon & Bonini '58)
- *mathematic idea:* *add more noise!* (Kesten '73)

$$x(t+1) = x(t)\left[1 + b(t)\right] + a(t)$$

  - b, a positive independent random variables
  - $a(t)$ acts as "effective repulsion" from zero (Sornette & Cont '97)

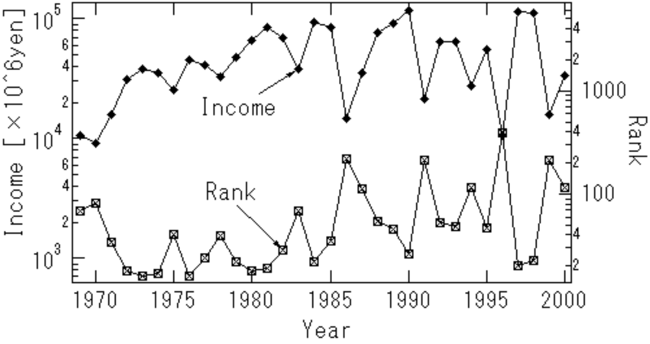
- *practical idea:* fit parameters (Takayasu et al '04)

$$x(t+1) = \alpha(t)\lambda(t,x)\,x(t) + a(t)$$

  - $[1+b] \to \lambda(x,t)$: growth depends on size
  - estimation from $\ln R(t) = \ln\{x(t+1)/x(t)\}$ with standard deviation $\sigma(x)$

Comparison with real company data

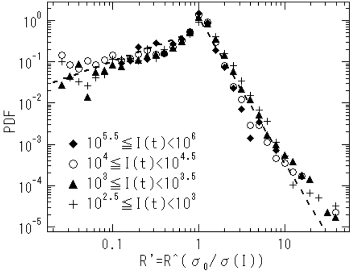
- Takayasu et al '04: income of 15.000 US and 15.000 non-US comp., 80.000 Japanese comp. (income > 40 Mio Yen), before tax



(Takayasu et al 2004)

$$x(t+1) = \alpha(t)\lambda(t,x)\,x(t) + f(t)$$

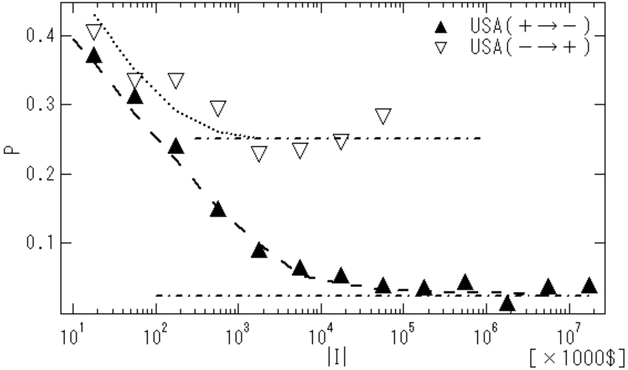
- $\lambda(x,t)$: growth depends on size
  - estimation from log growth rate: $\log R(t) = \log x(t+1) - \log x(t)$
    with standard deviation $\sigma(x)$
    for large x: σ_0, $f(t)/x$ negligible
    scaling by means of normalized growth: $R^{\sigma(x)/\sigma_0}$



(Takayasu et al 2004)

- $\alpha(t)$ either $+1$ (growth) or (-1) (slump)
  prob. determined empirically from large $|x(t)|$

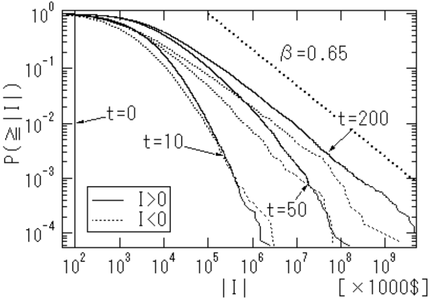
$$\alpha(t) = \begin{cases} 1 & \text{with prob. } 0.97\ (x(t) > 0),\ \ 0.75\ (x(t) < 0) \\ -1 & \text{with prob. } 0.03\ (x(t) > 0),\ \ 0.25\ (x(t) < 0) \end{cases}$$



(Takayasu et al 2004)

Forecast by means of Monte Carlo Simulations

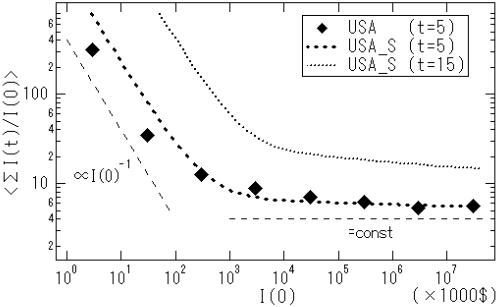
- initial state: 6.000 companies, $x_i(0) = 100$
  coefficients estimated from real data
- $t = 50$: qualitative agreement with real distribution (US)
  with constant growth rate distribution: firms income will keep growing for more than 100 years



(Takayasu et al 2004)

Investment Strategies?

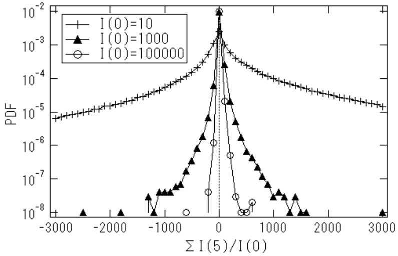
- normalized cummulative income for 5 years: $I = \sum_{n=1}^{5} x(n)/x(0)$

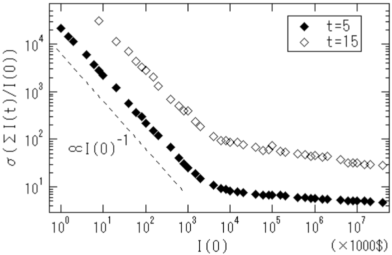


(Takayasu et al 2004)

- for $x(0) > 10^6$\$: $I \propto x(0) \Rightarrow$ invest in small firms?

- small firms: large growth rates, but also large variances (notice the asymmetric distribution)

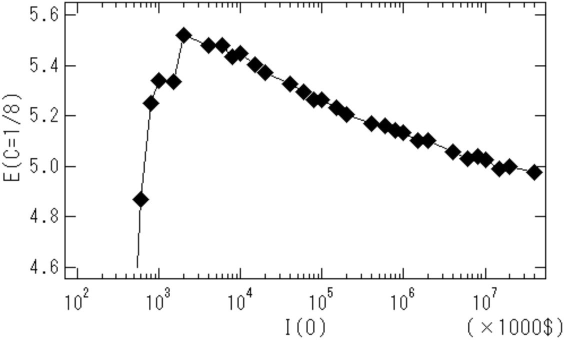




(Takayasu et al 2004)

- investment strategy: tradeoff between profits and risks
  investment efficiency: relation between $\langle I \rangle$ and $\sigma(I)$

$$E(c, x(0)) = \left\langle \sum \frac{x(5)}{x(0)} \right\rangle - c\,\sigma\left(\sum \frac{x(5)}{x(0)}\right)$$



(Takayasu et al 2004)

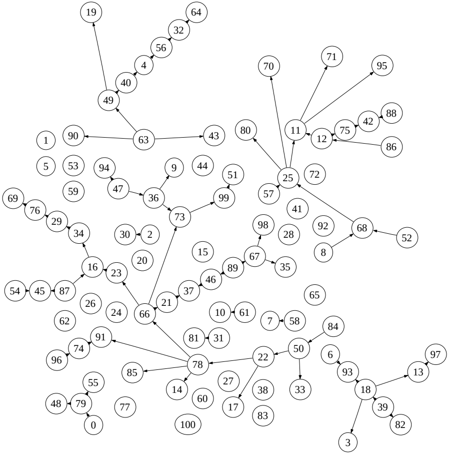
Growth through Network Effects

- $\dot{x}_i = \mathcal{F}_i = f(x_j, x_k) + \dots$

$$\frac{dx_i}{dt} = \sum_{j=1}^{N} c_{ij} x_j - \Phi x_i \qquad \text{(Jain/Krishna '98, '01)}$$

  - $c_{ij} \in \{0, 1\} \Rightarrow$ represents a directed network
    j catalyzes the growth of i, link probability p
    i is connected to $m = p(N-1)$ other companies (on average)
- *two time scales:*
  company growth (fast), network dynamics (slow)
- *assumption:* extremal dynamics $\Rightarrow$ minimum performance threshold

- **Questions**:
  - Under which conditions do companies survive?
  - Which structures of innovation networks emerge?
  - What happens if selection pressure is increased?
- **Results of computer simulations**:
  Emergence of a core of *cooperative* companies, and a *parasitic* pheriphery, considerable crashes and recovery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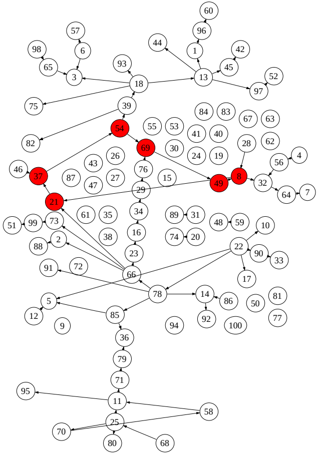
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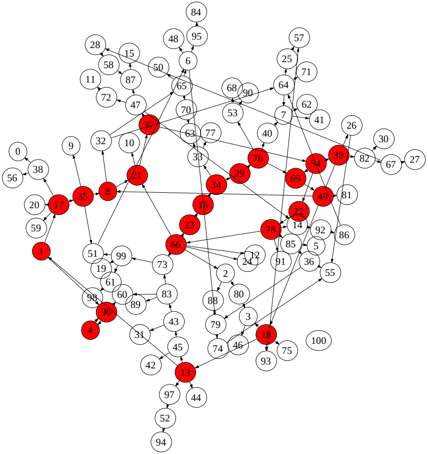
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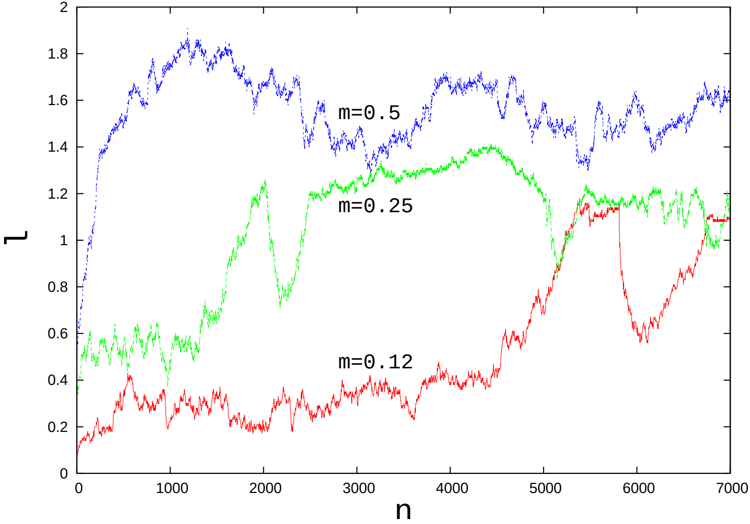
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3000
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7000
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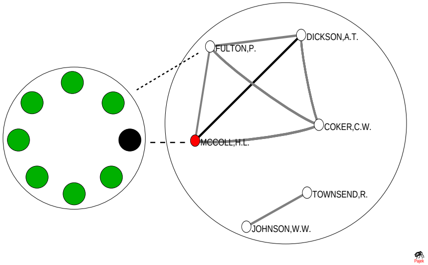
m=0.5
m=0.25
m=0.12

Get connected

- N agents with $\theta_i \in \{-1, +1\}$; ruling opinion $\theta_G = +1$
- CEO proposal $\Rightarrow$ N_+ supporters, N_- objectors
- additional (outside) ties between board members $\Rightarrow$ weight J_{ij}

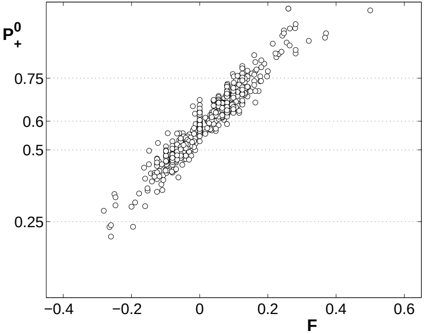


(Battiston et al 2003)

Model of board decisions (Battiston et al, 2003)

- probability of agent i to approve CEO proposal ($\theta_G = +1$):

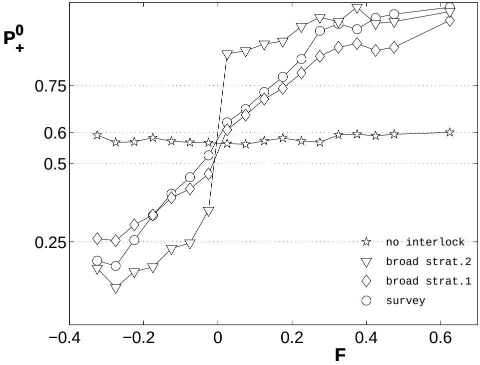
$$p = \frac{1}{1 + \exp\{-2\beta h_i(t)\}} \ ; \quad h_i(t) = \sum_{j \in NN} J_{ij}\theta_j$$



influence of lobby:
additional force
$F \sim \sum J_{ij}\theta_i(t = 0)$
$\Rightarrow$ *minority of well-connected members can drive the majority's decision*

The importance of speaking first

- instead of random sequential update: one at a time memory length $\gamma \Rightarrow h_i^\star = (1-\gamma)h_i + \gamma J_{ij}\theta_j$



(Battiston et al 2003)

Formation of Hierarchies

- complex system ("company"): various organizational hierarchies
  global aim: increase of "productivity" (utility, fitness, ...)

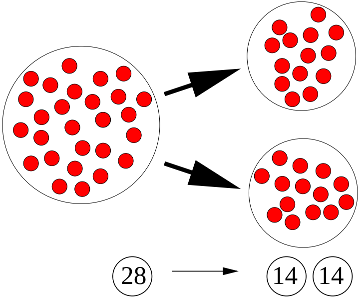
Simple, but illustrative model of Drossel '99:

- productivity of 1 unit at (lowest) level 1: $P_1(1)$ (negligible)
  productivity of N interacting units at (lowest) level 1: $P_1(N)$
  - *increases* with interaction possibilities: $P \sim N(N-1)$
  - *decreases* with costs of interaction (e.g. transportation costs)
    if system size increases linearly with N, then $P \sim -N(N-1)\,N$

$$P_1(N) = \left[g_1 N - c_1 N^2\right](N-1)$$

  - maximum size: $P_1(N) \to 0$: $N_{\max} = g_1/c_1$

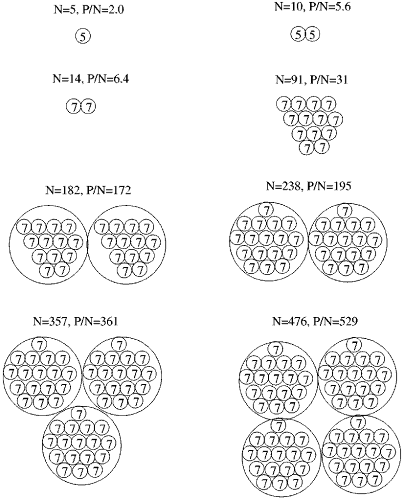
- maximum productivity per unit: $P_1(N)/N \to \max$
  $\Rightarrow$ optimal size $N_{\rm opt} = (g_1 + c_1)/2c_1$
- reasons to split into $N = N' + N''$ if $P_1(N) < P_1(N') + P_1(N'')$
  most profitable split for total productivity: $N' \approx N''$
  $\Rightarrow$ critical ("split") size: $N_{\rm crit} = 2(g_1 + c_1)/3c_1 = 4/3\ N_{\rm opt}$



- so far: optimization of *global* productivity
  - realistic system: cannot probe all possible configurations
    no breaking/rearrangement of large number of connections
  - growing systems: more likely follow established pathways
    ⇒ search for "local" optima (rather than global optima)
    ⇒ different set of growth rules lead to high productivity

Example 1:

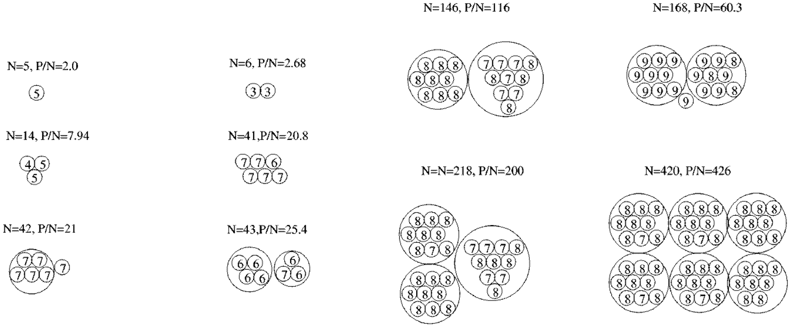
1. simultaneous formation of isolated groups until P_1/N decreases
2. groups interact → formation of supergroups until P_2/N decreases
3. supergroups interact → formation of super-supergroups until P_3/N decreases
4. groups at level $k-1$ can still grow further if this increases productivity at level k


N=5, P/N=2.0
N=10, P/N=5.6
N=14, P/N=6.4
N=91, P/N=31
N=182, P/N=172
N=238, P/N=195
N=357, P/N=361
N=476, P/N=529

(Drossel '99)

Example 2:

1. level 1: add new units $\rightarrow$ (i) join existing groups if this increases productivity, OR (ii) form a new group with one of the units in other groups, as long as productivity increases
2. migration of units from other groups to newly formed one, as long as productivity increases
3. level k: split of groups from supergroups to form new supergroups, and migration of groups to other supergroups, as long as productivity increases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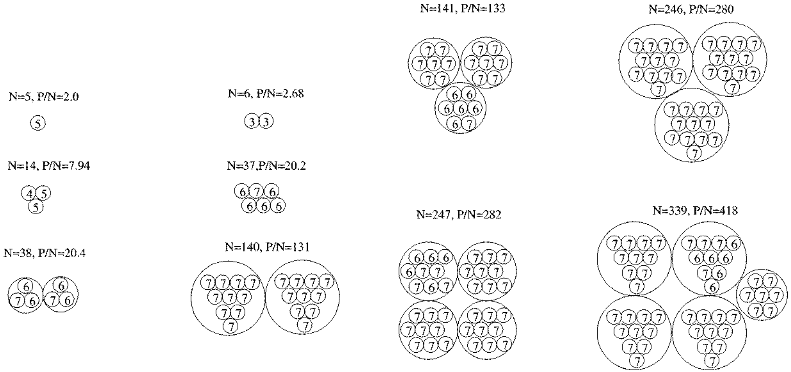
N=6, P/N=2.68
N=146, P/N=116
N=168, P/N=60.3
N=14, P/N=7.94
N=41,P/N=20.8
N=N=218, P/N=200
N=420, P/N=426
N=42, P/N=21
N=43,P/N=25.4

(Drossel '99)

Example 3: (Drossel '99)

1. add new units to a group until productivity decreases
2. split into two groups that grow until productivity decreases
3. rearrangement into three groups that grow until productivity decrease
4. split into two supergroups that grow until productivity decreases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N=6, P/N=2.68
N=14, P/N=7.94
N=37,P/N=20.2
N=38, P/N=20.4
N=140, P/N=131
N=141, P/N=133
N=247, P/N=282
N=246, P/N=280
N=339, P/N=418

(Drossel '99)

Result:

- complexity emerges: formation of hierarchies to optimize two contradicting requirements (benefit vs. costs of interaction)

Extensions:

- *heterogeneous agents:* no identical units, groups, ...
- explicite time dependence: "aging of groups"
- explicite dependence on distance, costs of migration
- dynamics of entry/exit: "birth" dependent on local conditions, "death" of units, groups, ...
- dependence on resources

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