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Multiplicative Models for Company Dynamics

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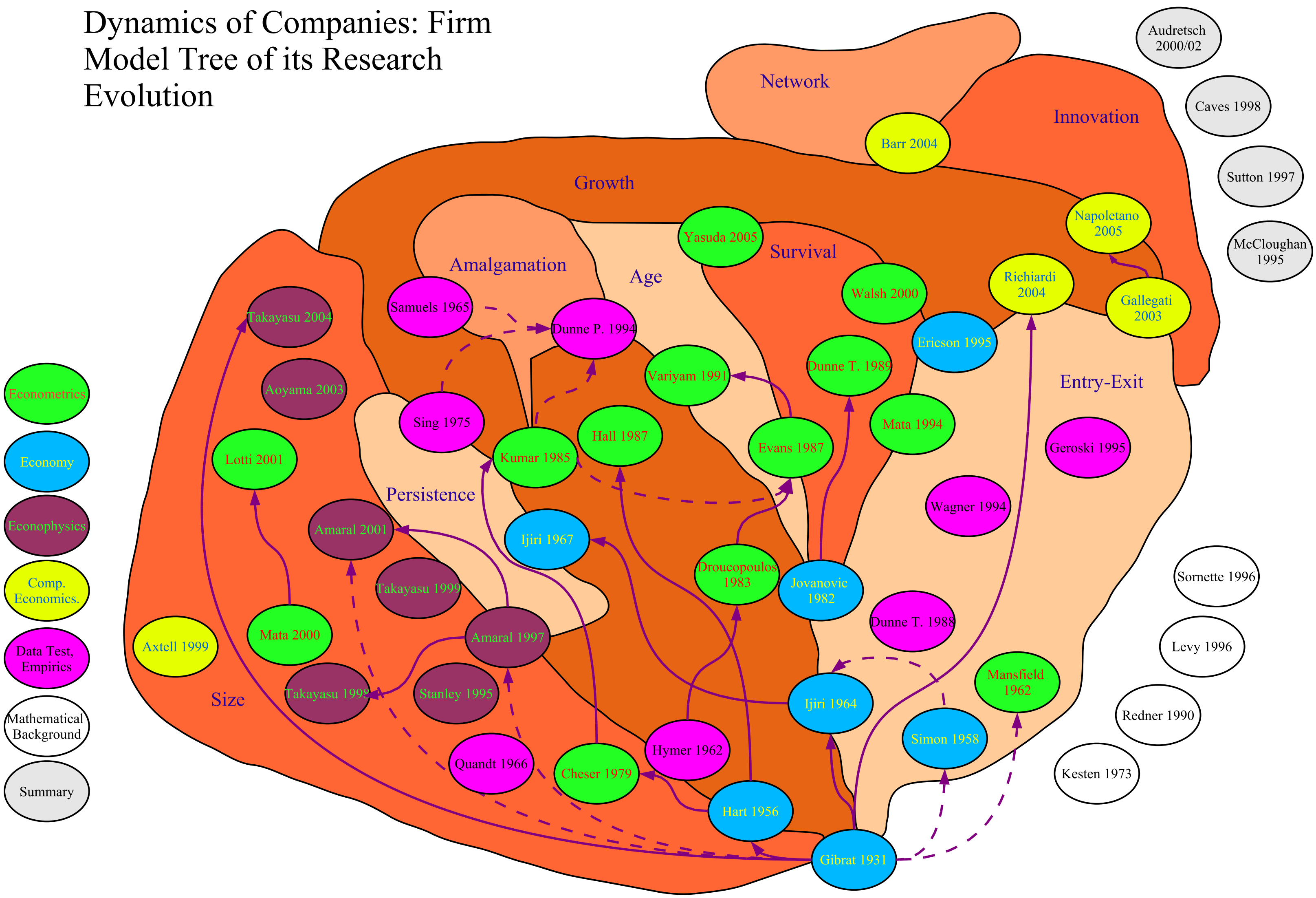
Thanks to ...

S. Jain, L. Amaral, H. Takayasu, A. Seufert, N. Bürkler...

Dynamics of Companies: Firm Model Tree of its Research Evolution



Company Growth

- set of companies: $i = 1, ..., N$
  - $x_i(t)$: company "size" (income, output, employees, ...)
  - growth rate: $dx_i/dt = \mathcal{F}_i(?)$
- $\mathcal{F}_i(t)$ with $\langle \mathcal{F}_i(t) \rangle = 0$, $\langle \mathcal{F}_i(t)\mathcal{F}_i(t') \rangle = S\delta_{ij}\delta(t-t')$

$$x_i(t + \Delta t) = x_i(t) + \sqrt{S\Delta t}\,\xi_i$$

  - growth as random walk (Bachelier, 1900)
- $\mathcal{F}_i = f(x_i) + ... \Rightarrow$ rest of this talk
  - independent growth, proportional to size (Gibrat, 1931)
- $\mathcal{F}_i = f(x_j, x_k) + ...$
  - growth through innovation networks

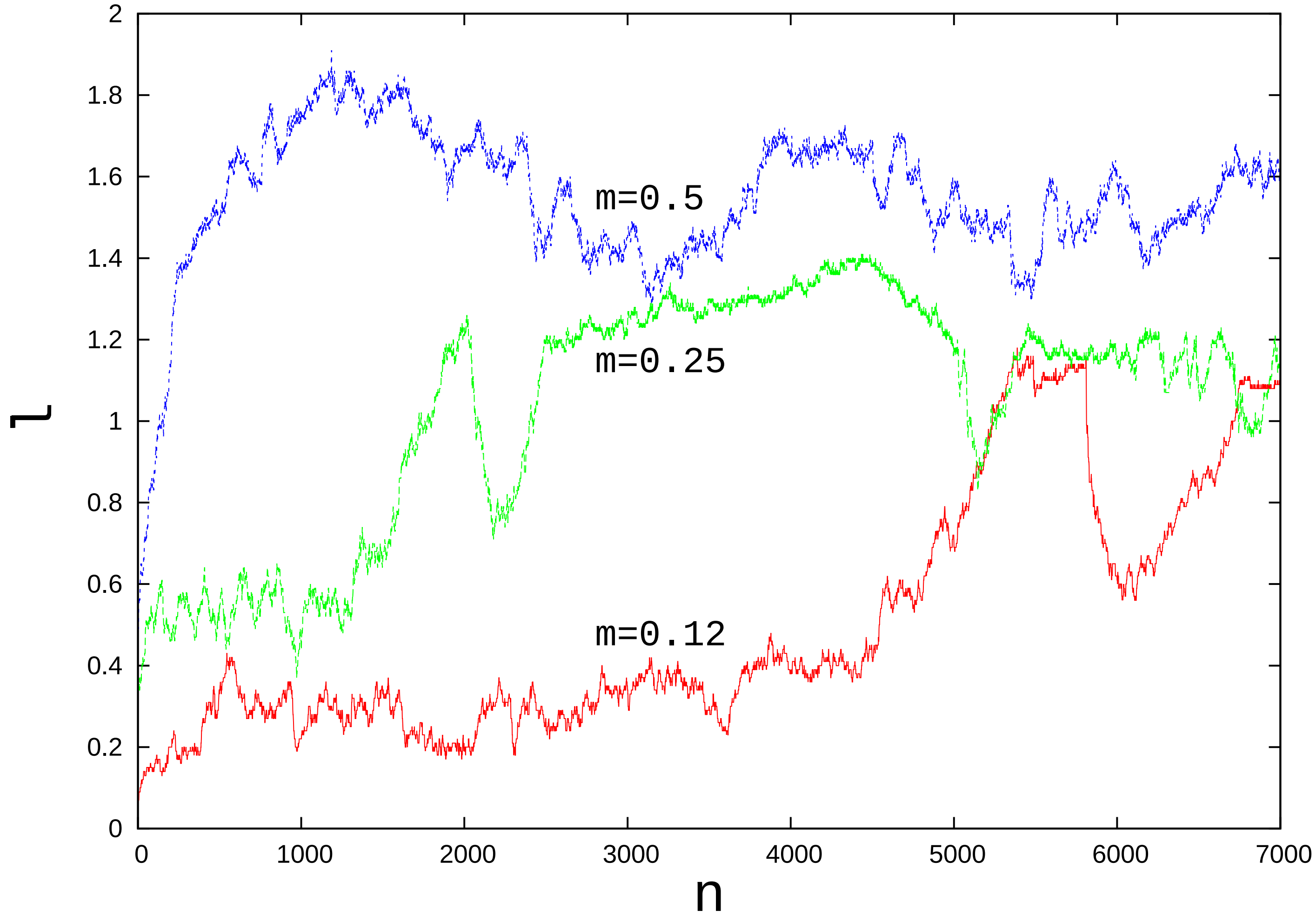
Growth through Network Effects

- $\dot{x}_i = \mathcal{F}_i = f(x_j, x_k) + ...$

$$\frac{dx_i}{dt} = \sum_{j=1}^{N} c_{ij} x_j - \Phi x_i \qquad (\text{Jain/Krishna}\ '98,\ '01)$$

  - $c_{ij} \in \{0, 1\} \Rightarrow$ represents a directed network
    - $\star$ j catalyzes the growth of i, link probability p
    - $\star$ i is connected to $m = p(N-1)$ other companies (on average)
- *two time scales:* company growth (fast), network dynamics (slow)
  assumption: extremal dynamics $\Rightarrow$ minimum performance threshold
- questions:
  - Under which conditions do companies survive?
  - Which structures of innovation networks emerge?
  - What happens if selection pressure is increased?

Result: Emergence of a core of *cooperative* companies, and a *parasitic* pheriphery, considerable crashes and recovery



Multiplicative Growth

- $\dot{x}_i = \mathcal{F}_i = f(x_i) + ... = b\, x_i$
  - "Law of proportionate growth" (Gibrat '30, '31; Sutton '97)
  - no interactions between firms

$$x_i(t+\Delta t) = x_i(t)\left[1 + b_i(t)\right]$$

- Assumptions:
  - $b_i(t)$: independent of i, no temporal correlations (random noise)
  - *growth "rates":* $R(t) = x(t+1)/x(t),\; t \gg \Delta t,\; \ln(1+b) \approx b$

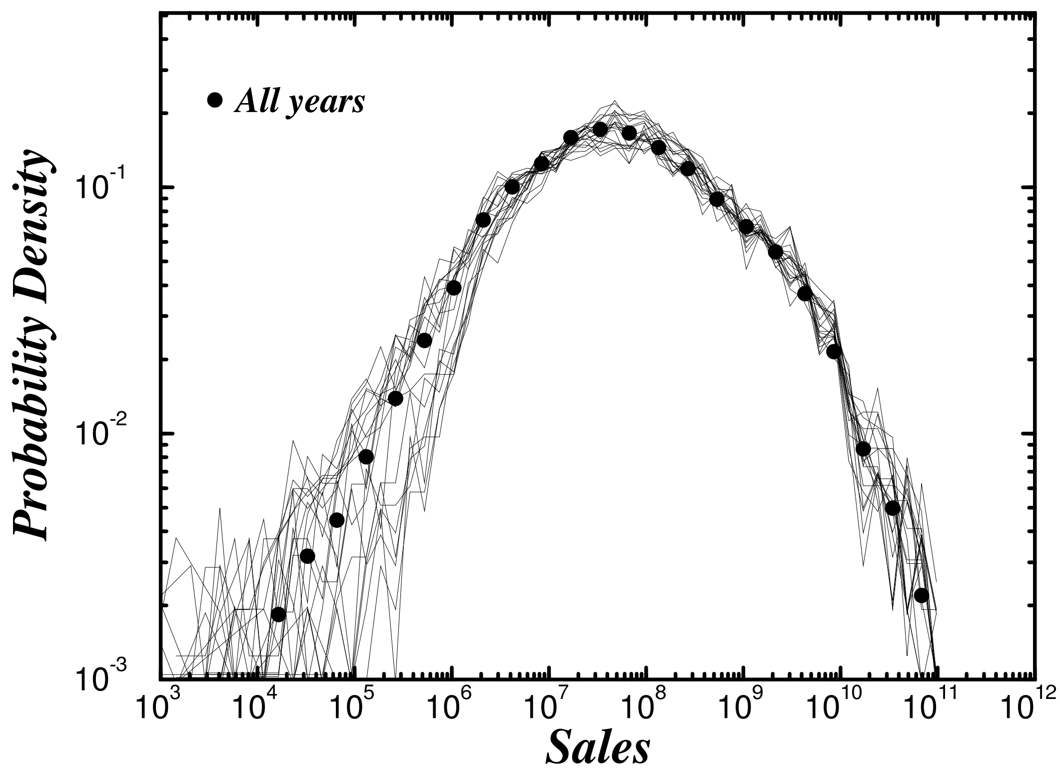
$$\ln R(t) = \sum_{n=1}^{t} b(n)$$

$\Rightarrow$ random walk for $\ln R(t)$ $\Rightarrow$ log-normal distribution for $x_i(t)$

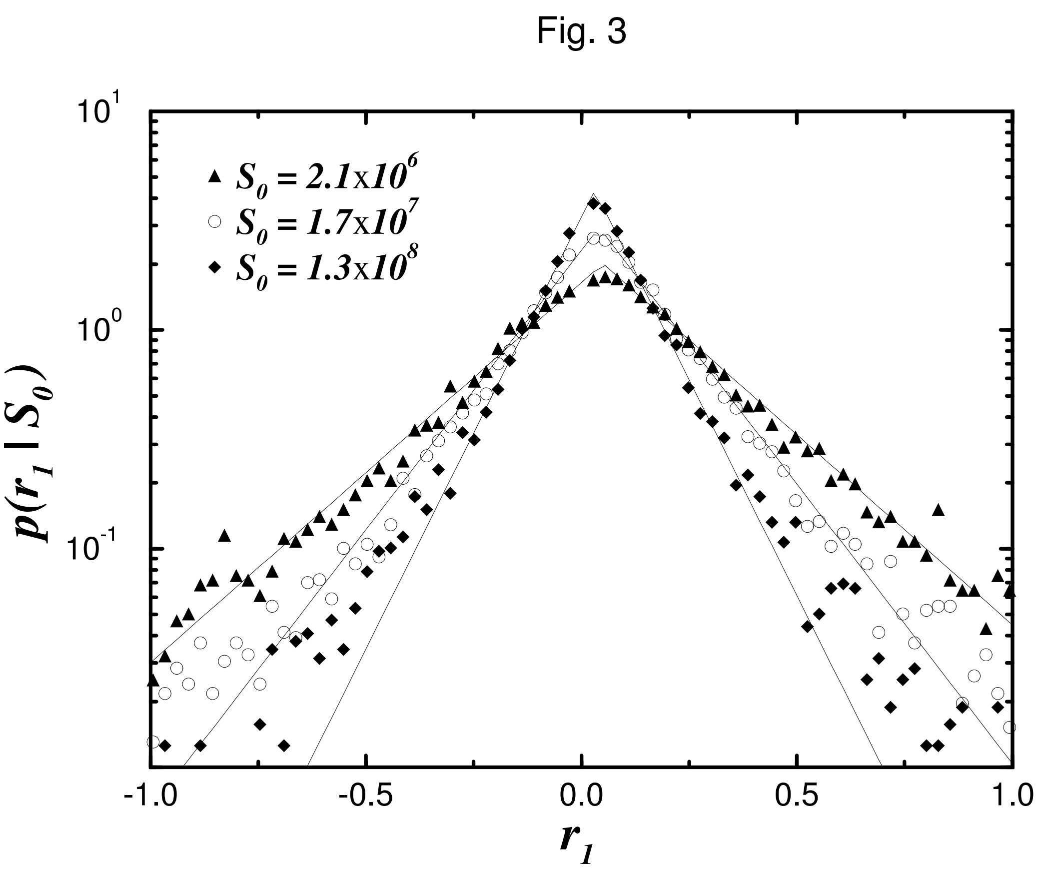
- empirical evidence?

➤ Empirical distribution of company sizes (1974-1993) (Amaral et al, 1997) $\Rightarrow$ log-normal distribution

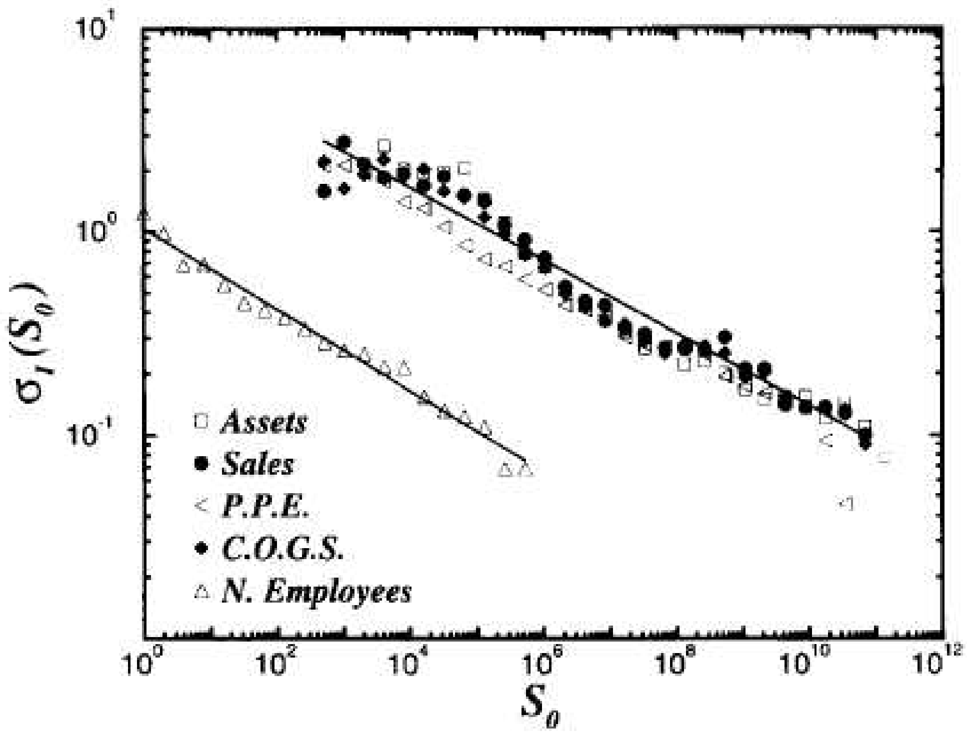
Fig. 2(a)



➤ Empirical distribution of growth rates (Amaral et al, 1997)
$\Rightarrow$ depend on size, tent-shape exponential distribution



➤ Empirical distribution of standard deviation of growth rates (Amaral et al, 1997) $\Rightarrow$ depend on size, power-law distribution



Stylized facts:

- log-normal distribution of company sizes

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma\, x} \exp\left[\frac{(-\ln x - \mu)^2}{2\sigma^2}\right]$$

- exponential growth ratio distribution

$$P(r_1|x_0) = \frac{1}{\sqrt{2}\,\sigma_1(x_0)} \exp - \left[\frac{\sqrt{2}\,|r_1 - \bar{r}_1(x_0)|}{\sigma_1(x_0)}\right]$$

- power-law distribution of the standard deviations

$$\sigma_1(x_0) \sim x_0^{-\beta}\,; \quad \beta < 0.5$$

Explanations:

- correlations in the growth rates
  company is attracted to an “optimal size”

$$\frac{x_{t+\Delta t}}{x_t} = \begin{cases} k e^{\varepsilon_t}, & x_t < x^* \\ \frac{1}{k} e^{\varepsilon_t}, & x_t > x^*, \end{cases}$$

- growth depends on properties of management hierarchies
  n levels, z units, decisions on higher level are followed with prob π

$$\beta = \begin{cases} -\ln(\pi)/\ln(z) & \text{if} \quad \pi > z^{-1/2} \\ 1/2 & \text{if} \quad \pi < z^{-1/2} \end{cases}$$

  - β decreases in time $\Leftrightarrow$ companies better coordinated

Further Improvements of Gibrat's Model

- *economic idea:* simple entry dynamics (Simon & Bonini '58)
- *mathematic idea: add more noise!* (Kesten '73)

$$x(t+1) = x(t)\big[1 + b(t)\big] + a(t)$$

  - b, a positive independent random variables
  - $a(t)$ acts as "effective repulsion" from zero (Sornette & Cont '97)

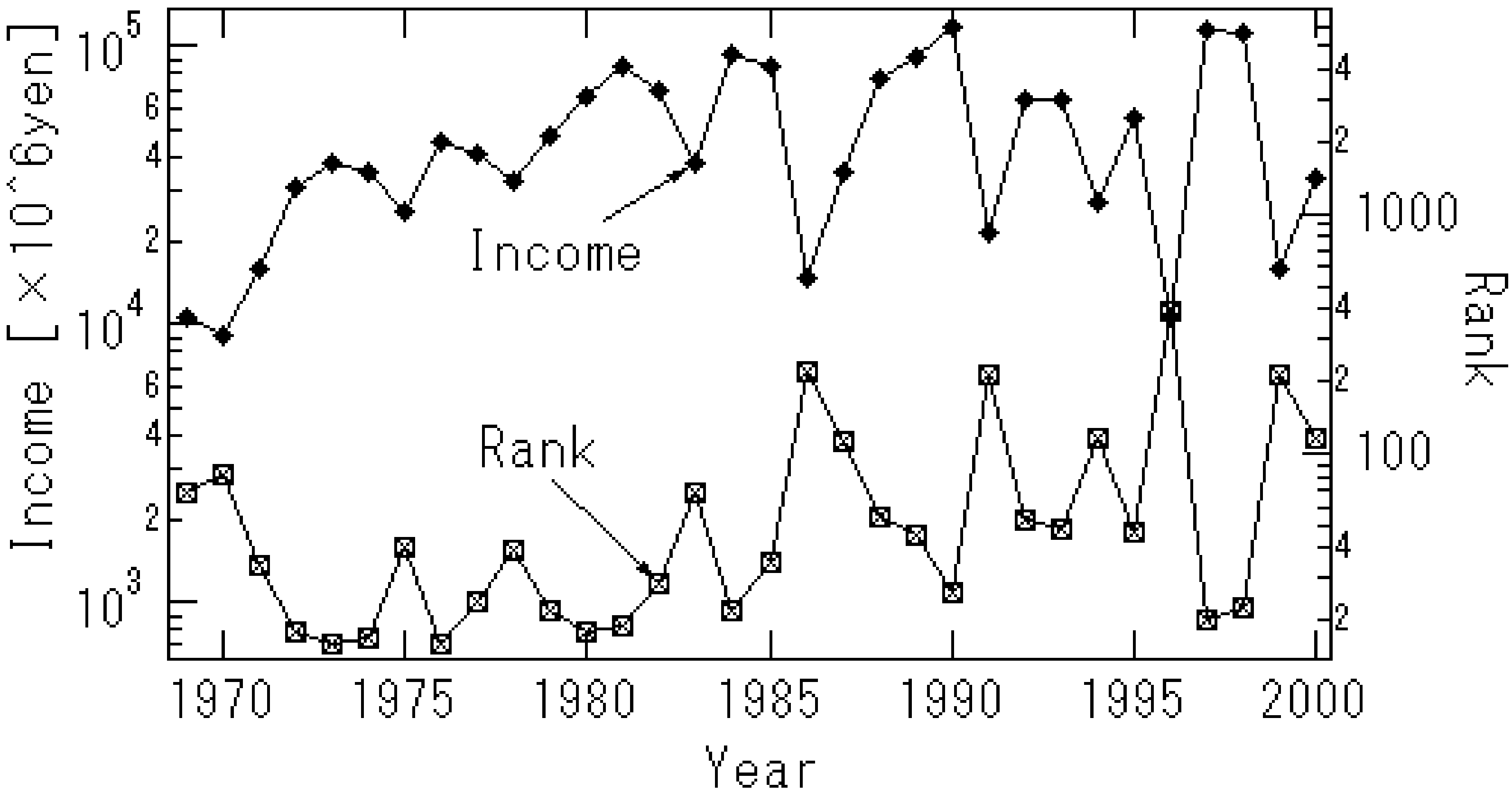
- *practical idea:* fit parameters (Takayasu et al '03)

$$x(t+1) = \alpha(t)\lambda(t,x)\,x(t) + a(t)$$

  - $[1+b] \rightarrow \lambda(x,t)$: growth depends on size
  - estimation from $\ln R(t) = \ln\{x(t+1)/x(t)\}$ with standard deviation $\sigma(x)$

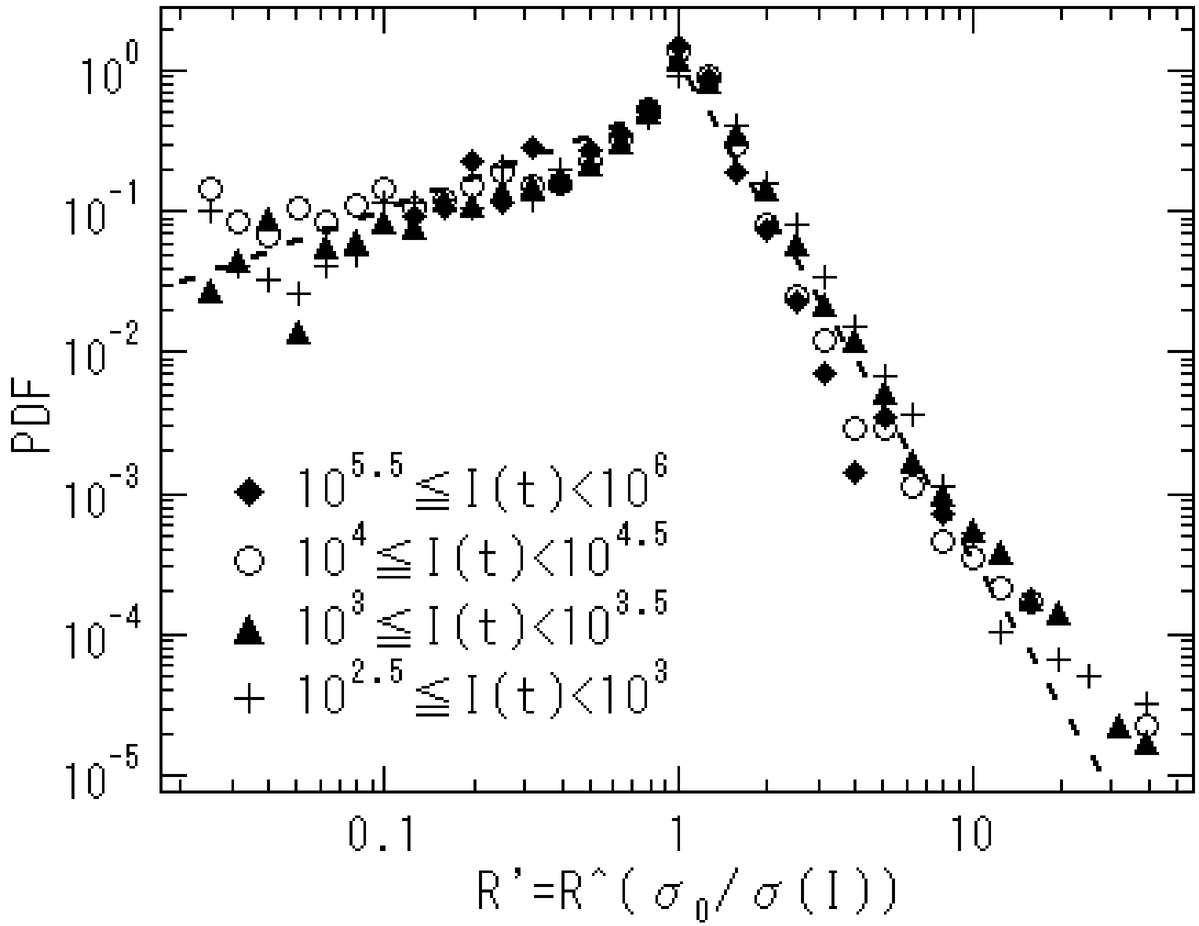
Comparison with real company data

- Takayasu et al ’03: income of 15.000 US and 15.000 non-US comp., 80.000 Japanese comp. (income > 40 Mio Yen), before tax



Estimation of $\ln R(t)$, $\sigma(x)$

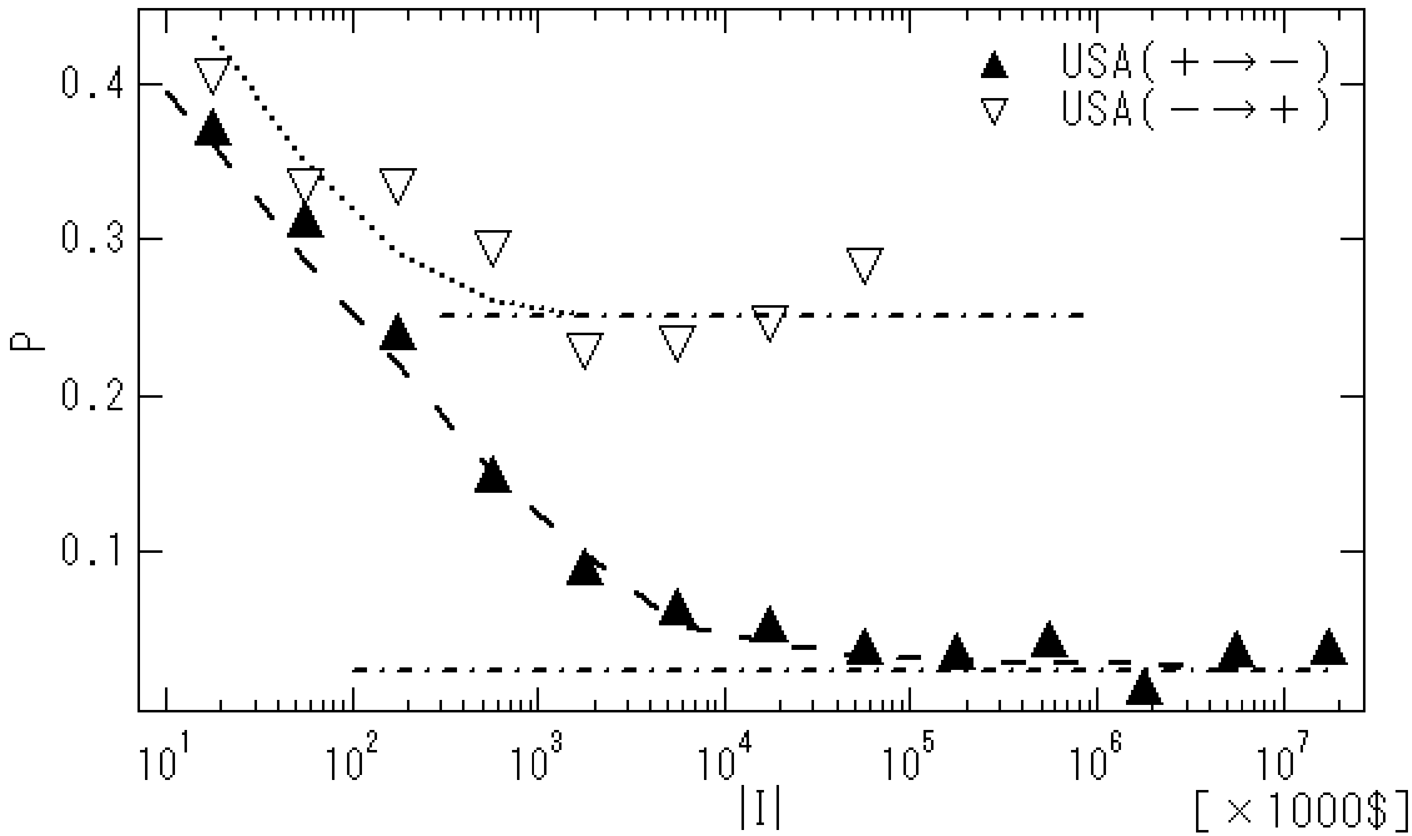
- for large x: σ_0, $f(t)/x$ negligible
  scaling by means of normalized growth: $R^{\sigma(x)/\sigma_0}$



(Takayasu et al ’03)

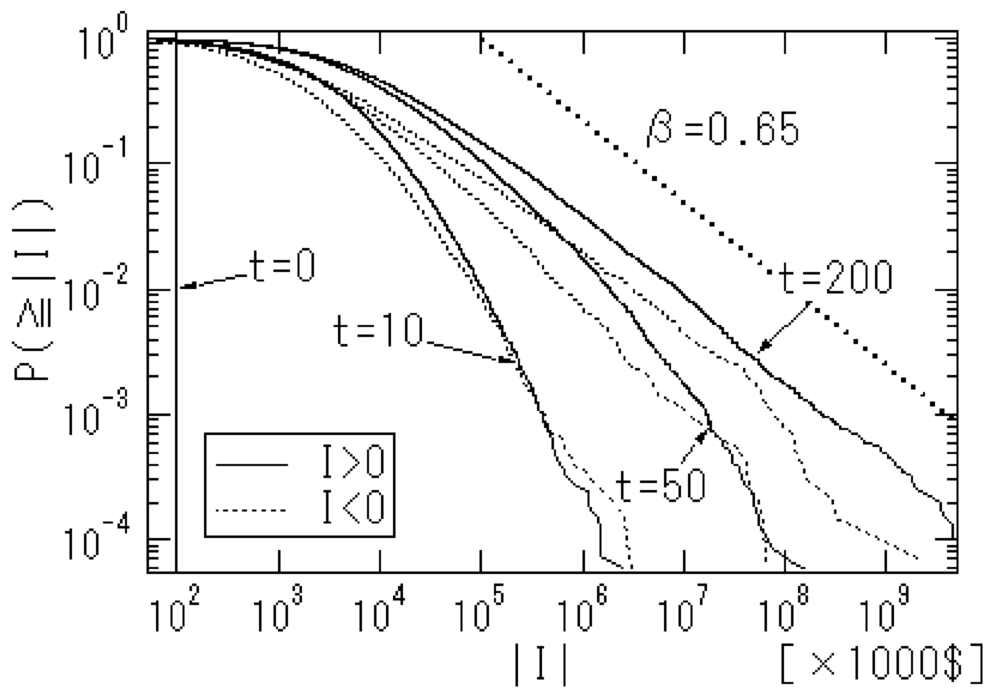
- $\alpha(t)$ either +1 (growth) or (-1) (slump)
  prob. determined empirically from large $|x(t)|$

$$\alpha(t) = \begin{cases} 1 & \text{with prob. } 0.97 \ (x(t) > 0), \ \ 0.75 \ (x(t) < 0) \\ -1 & \text{with prob. } 0.03 \ (x(t) > 0), \ \ 0.25 \ (x(t) < 0) \end{cases}$$



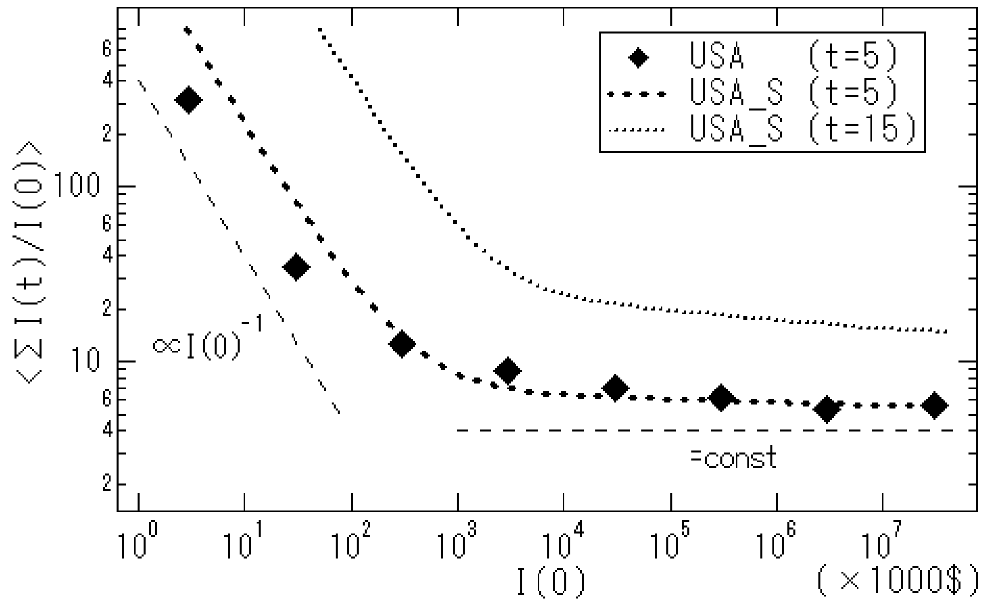
Forecast by means of Monte Carlo Simulations

- initial state: 6.000 companies, $x_i(0) = 100$
  coefficients estimated from real data
- $t = 50$: qualitative agreement with real distribution (US)
  with constant growth rate distribution: firms income will keep growing for more than 100 years



Investment Strategies?

- normalized cummulative income for 5 years: $I = \sum_{n=1}^{5} x(n)/x(0)$



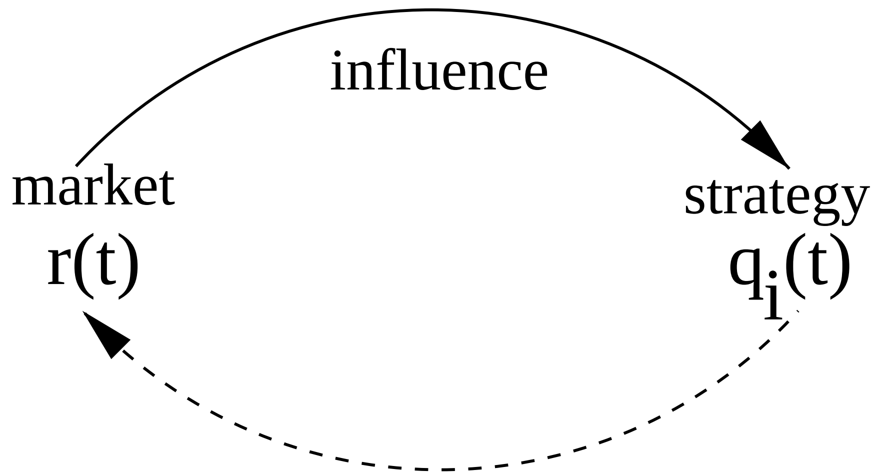
- for $x(0) > 10^6$\$: $I \propto x(0) \Rightarrow$ invest in small firms?

Topics for Future Investigations

1. Growth strategies

- companies with growth strategies: $b(t) \rightarrow r(t)q_i(t)$

$$x_i(t+\Delta t) = x_i(t)\Big[1 + r(t)q_i(t)\Big] + a$$



- $r(t) < 0 \Rightarrow q_i(t) \rightarrow q_{\mathrm{min}}, \quad r(t) > 0 \Rightarrow q_i(t) \rightarrow q_{\mathrm{max}}$
  - Can company predict $r(t)$? Can it adjust $q_i(t)$ fast enough?

2. Stochastic Models with interactions

➤ combining multiplicative growth models and network models

$$x_i(t+1) = x_i(t)\left[1 + b_i(t) + \sum_{j=1}^{N} c_{ij} x_j\right] + \mathcal{G}(t) - \mathcal{L}(t)$$

- network dynamics: $c_{ij} \Rightarrow c_{ij}(t, \boldsymbol{r})$
  represents spatial interaction, cooperation, spread of innovations, ...
- “catalytic´´ growth: $b_i(t) = r(t) q_i(t)$
  represents strategic decisions, organizational efficiency, ...
- $\mathcal{G}(t)$, $\mathcal{L}(t)$ represent external conditions (gov, environment, ...),
  global couplings (limited ressources, ...)

Conclusions

- *company growth:* field of applications for stochastic processes
  - network effects? ⇒ innovation economics, input from physics?
- *multiplicative growth models:* are successfully applied to real data
  - stylized facts reproduced
  - conceptual drawbacks: random growth ⇔ firms as profit maximizers respond similarly to changing market conditions
- *micro models ??*
  - hierarchical-tree model (Amaral et al, 1997)
  - multi-agent models (Axtell, ...)
  - heterogeneity? market ecology of different organizational forms