How damage diversification can reduce systemic risk

Rebekka Burkholz,[*] Antonios Garas,[†] and Frank Schweitzer[‡]
ETH Zurich, Chair of Systems Design, Weinbergstrasse 56/58, 8092 Zurich, Switzerland
(Received 3 March 2015; revised manuscript received 15 January 2016; published 22 April 2016)

We study the influence of risk diversification on cascading failures in weighted complex networks, where weighted directed links represent exposures between nodes. These weights result from different diversification strategies and their adjustment allows us to reduce systemic risk significantly by topological means. As an example, we contrast a classical exposure diversification (ED) approach with a damage diversification (DD) variant. The latter reduces the loss that the failure of high degree nodes generally inflict to their network neighbors and thus hampers the cascade amplification. To quantify the final cascade size and obtain our results, we develop a branching process approximation taking into account that inflicted losses cannot only depend on properties of the exposed, but also of the failing node. This analytic extension is a natural consequence of the paradigm shift from individual to system safety. To deepen our understanding of the cascade process, we complement this systemic perspective by a mesoscopic one: an analysis of the failure risk of nodes dependent on their degree. Additionally, we ask for the role of these failures in the cascade amplification.

DOI: 10.1103/PhysRevE.93.042313

I. INTRODUCTION

In the course of globalization and technical advancement, systems become more interconnected and system components more dependent on the functioning of others [1–3]. In particular for socioeconomic networks [4] and financial networks [5,6] we observe an increase in coupling strength and complexity at the same time. Examples are global supply chains, but also technical systems, as, e.g., power-grids in the USA and Europe [7].

Increased dependence, under normal conditions, has advantages for the efficient operation of a system. It also leads to an increased risk diversification of its components, since the dependence on single other components is reduced. According to classical risk management theories [8], a higher connectivity thus decreases the vulnerability of the system as a whole, even with respect to cascading failures. Several authors, e.g., Refs. [9–13], pointed out that a higher connectivity can also increase systemic risk, i.e., in our context the expected cascade size.

Although these findings are partially model-dependent, they can be explained intuitively by a basic tradeoff between diversification and system connectivity that is reflected in the risk exposures. In fact, most of these works are based on Watts's cascade model [14] that introduces this tradeoff, and can be interpreted as a study of simple diversification strategies of system components or agents that are represented as nodes in a network. Links between two nodes stand for a dependency between them and they are called network neighbors. Here, we call this model the *exposure diversification* approach (ED), since increased diversification of system components reduces their exposure to the failure of single other components. But at the same time, it increases the connectivity of the overall system. Especially, if well connected components, so called *hubs*, fail they affect a high number of other nodes. Even if the inflicted damage is only little, if it is enough to trigger a few other failures, the cascade might continue and span a large fraction of the entire system.

This implies that hubs, because of their large number of neighbors, considerably affect the network in case of failure. Consequently, policy discussions and risk reduction strategies center around the question of how to prevent the failure of hubs, e.g., by increasing their robustness (i.e., relative capital buffers in finance [13,15] or immunization in epidemiology [16]).

Here, we test in a generic modeling approach a way to complement such regulatory efforts and strategies by the mitigation of the *impact* of the failures of well-connected nodes. Thus, we contrast the ED approach with a scenario in which the *impact* of a failing node (rather than its *exposures*) is diversified. This *damage diversification* (DD) approach has the potential to reduce systemic risk significantly, since it counterbalances the failure amplification caused by hubs. For instance, in a financial network this approach would correspond to a policy where each financial institution is only allowed to get into a limited amount of debt.

We are aware that our approach is abstract, and it is based on simplifying assumptions that can deviate from real-world application scenarios. For instance, insolvency proceedings of financial institutions are complicated and last for longer time periods. Not all concerned parties are affected in the same way, and different financial products have different consequences for the counterparties. Regarding epidemic spreading, people respond differently to infections, have various incubation times, and infections are often not transferred just by contact. Also many ecological factors influence the propagation of forest fires. Cascades generally do not proceed without any (human) interventions and, often, the considered network structures are not static.

Without question, adequate simplifications of the real world lie at the core of every model. As a matter of fact, one simplification is introduced by the approximation of complicated interaction patterns by a simple network. The topology of such a network is then one of the main influencing factors of systemic risk.

[*]rburkholz@ethz.ch
[†]agaras@ethz.ch
[‡]fschweitzer@ethz.ch

Research that is concerned with monitoring systemic risk focuses on the inference of such network structures by empirical observations. The studied networks vary in properties like their degree distribution, clustering coefficient, and mesoscopic structures. Despite their high relevance for the application scenarios at hand, such works rarely allow for the identification of the main drivers of systemic risk that are common to a larger class of cascade processes.

Such classes are usually studied in a generic (and simplifying) modeling approach. Often, random graph ensembles are considered and average properties are measured to form an expectation of the cascade size under network uncertainty, or to summarize results for a big class of different networks.

In this paper, we follow a generic modeling approach for systemic risk and we are especially interested in the role of heterogeneous degrees and thus heterogeneous diversification strategies. Thus, the configuration model [17,18] is the most appropriate choice to generate random graph ensembles, as the resulting network distribution maximizes the entropy under the constraint of a given degree sequence.

We present simulations as well as analytic derivations for network ensemble averages. The derivations are valid in the limit of infinite network size, where two quantities on the system level are given: (a) the degree distribution, which defines the number of direct neighbors of a node, and thus limits their respective diversification strategies; and (b) the distribution of robustness, which is later defined by the failure threshold. The analytic method (also known as heterogeneous mean field or branching process approximation) has been derived for the ED approach [12]. But, it does not capture processes where the impact of a failing neighbor depends on its specific properties (e.g., its degree or robustness) as it is required for the treatment of the DD variant. Therefore, we extend the branching process approximation to the latter case and generalize it for the application to weighted random network models. The weight statistics could be deduced from data, taking also into account different properties of neighboring nodes, or from specific model assumptions. This way, we generalize the analytic treatment to match application scenarios better.

Despite our generic modeling ansatz, our approach can be of relevance to practitioners in cases when network structures change rather quickly, which introduces network uncertainty, or in cases when simulations are computationally demanding. In the case of the latter, an analytic branching process approximation is still feasible as a proxy for the average cascade size. However, it is important to note that many complicated microscopic network structures are not considered in such a systemic risk analysis, although they might have a high impact on the results. While studies for the ED variant suggest that clustering does not seem to bias a systemic risk analysis, a large mean intervertex distance can be problematic [19]. In addition, strong degree-degree correlations can also be of high relevance [20,21].

Our main methodological contribution is the study of generic cascade models that do not aim at monitoring systemic risk, but seek instead for system design principles that can reduce systemic risk. Especially, our approach allows us to deepen the understanding of the role of different risk diversification strategies. In fact, it translates a recent paradigm shift to an analytic setting. The focus has been shifted from individual or componentwise risk assessments to systemic risk analysis. Thus, the question of the failure risk of single components is complemented by the question for the consequences of their failure for the rest of the system. A systemic risk analysis is often accompanied by the identification of so-called system-relevant nodes. But this analysis is based on the assumption that various components can have a different impact on the system stability. This is considered in our framework, as failing nodes can cause different damage to their network neighbors.

As a result of the tradeoff between system connectivity and risk diversification, an increased diversification does not need to reduce the failure risk, neither of the system nor its components. Increased diversification does not even need to decrease the failure risk of a component despite the fact that it reduces systemic risk.

Because of this, we accompany our systemic risk measure on the macro level, the average cascade size, with a comparison of failure probabilities of nodes with different diversification strategies on the meso level. The nodes' systemic relevance can then be identified on the basis of their cascade amplification role.

II. MODELING EXPOSURE VERSUS DAMAGE DIVERSIFICATION

In a weighted (directed) network with N nodes a link with positive weight $w_{ji} \geqslant 0$ between two nodes j and i, represents an exposure of i to its network neighbor j. Each node j can fail either initially or later in (discrete) time t because of a propagating cascading process. Its (binary) state then switches from $s_i(t) = 0$ (okay) to $s_i(t) = 1$ (failed), without the possibility to recover.

If the node j fails, its neighbor i faces the loss w_{ji}. The total amount of i's losses sums up to

$$L_i(t+1) = \sum_j w_{ji} s_j(t). \tag{1}$$

If L_i exceeds the threshold θ_i (i.e., $L_i \geqslant \theta_i$), the node i fails as well. Hence, θ_i expresses the *robustness* of node i. In this way a cascade of failing nodes develops over time, which can even span the whole network.

We call L_i also *load* instead of, e.g., losses, and w_{ji} also damage or impact of node j on node i to indicate that we do not necessarily have financial systems in mind, and instead follow a generic modeling approach where we focus on common principles in cascade models. In fact, w_{ij} would often not correspond to a loss, but to a loss relative to a node's equity in a financial setting. For an in-depth explanation of how to link this model to a balance-sheet approach, we refer the interested reader to Refs. [22,23]. Here, we just borrow the intuition for cascading losses in a network.

We measure the cascade size by the final fraction of failed nodes,

$$\rho_N = \lim_{t\to\infty} \frac{1}{N} \sum_{i=1}^{N} s_i(t), \tag{2}$$

when no further failures are triggered. Any cascade stops after at most N time steps, since at least one node needs to fail at a time to keep the process going.

The cascade dynamics are fully deterministic for given thresholds and weights on a fixed network. Still, many systems do not remain constant over time and may also fluctuate by their exposure to large cascades. Consequently, it is reasonable to quantify the risk of large cascades with respect to macroscopic distributions that allow for microscopic variations of the weighted network and the thresholds. The average cascade size with respect to these distributions then defines our measure of systemic risk.

In such a setting, we study the influence of two diversification variants. The difference between the ED and the DD approach is in defining the weights w_{ji}. Precisely,

$$w_{ji}^{(\mathrm{ED})} = \frac{1}{k_i}; \quad w_{ji}^{(\mathrm{DD})} = \frac{1}{k_j}, \tag{3}$$

for *exposure diversifications* (ED) and for *damage diversifications* (DD), respectively. Here, k_i denotes the degree of node i, i.e., the number of its neighbors.

In the ED case, every neighbor is treated identical; i.e., a failure of any neighboring node j exposes a node i to the same loss $1/k_i$. The higher the degree k_i of node i, the better it diversifies its total exposure of $\sum_{j\in \mathrm{nb}(i)} 1/k_i = 1$, where $\mathrm{nb}(i)$ denotes the neighbors of node i; Thus, in the ED case, single failures of neighbors j become less harmful to node i if it has a larger number of neighbors. On the other hand, the failure of a hub impacts many other nodes and is thus problematic from a system perspective.

In the DD case, the impact of a hub is effectively reduced. The failure of a hub j causes a total loss of $\sum_{i\in \mathrm{nb(j)}} 1/k_j = 1$, which reduces the impact on a neighboring node i to $1/k_j$ (instead of $\frac{1}{k_i}$ in the ED case, where normally $k_i \ll k_j$). Hence, better diversified nodes damage each of their neighbors effectively less in case of a failure.

We note that, in general, $w_{ji}^{(\mathrm{ED})} \neq w_{ij}^{(\mathrm{ED})}$ and $w_{ji}^{(\mathrm{DD})} \neq w_{ij}^{(\mathrm{DD})}$. Still, we can assume an undirected skeleton network, i.e., we do not need to distinguish between in-degrees and out-degrees. This differs from the usual approach for directed networks [11,22,24], where the neighbors whose failures impact a node are distinct from the ones who face a loss in case of the node's failure. In this case, one node is exposed to the other, but not vice versa. In our case, however, once there is a link between two nodes, each can impact the other, but the amount of the loss can be different. Still, this does not limit us to the study of undirected networks, since weights w_{ji} in one direction can also be set to zero ($w_{ji} = 0$). Instead, it allows us to model situations where some nodes are part of mutual dependencies as well as one-way exposures. Thus, our approach is a generalization. In the Appendix we provide an example for an exclusively directed network by defining the weights as net difference between weights in the DD approach: $w_{ji}^{(\mathrm{eff})} = \max(1/k_j - 1/k_i, 0)$.

We are interested in how the heterogeneity of such diversification strategies as ED and DD and the heterogeneity of thresholds impact systemic risk for large systems. In both model variants the diversification strategies are determined by the degrees of the nodes.

Consequently, we study the fraction of failed nodes as an average over a whole class of networks characterized by a fixed-degree distribution $p(k)$ and a fixed-threshold distribution $F_\Theta(\theta)$ in the limit of infinitely large networks ($N \to \infty$). The network generation method with fixed-degree sequence, which can be drawn from a degree distribution $p(k)$, is known as configuration model [17,18]. There, all possible network realizations are equally likely, but we condition on the property that the network is simple, i.e., it has no multiple edges or self-loops. This means in our numerical simulations we do not regard networks with multiple edges and self-loops similar to Ref. [25]. The thresholds are then assigned to nodes independently of each other, and independently of their degree according to $F_\Theta(\theta)$, although the independence of the degree is not a necessary assumption. In the next section, we will derive our formula for the final cascade size for a more general case, where the law $F_\Theta(\theta)$ for the random threshold Θ of a node can also depend on the node's degree k. So we have $F_{\Theta(k)}(\theta)$.

For the ED approach, the average fraction of failed nodes at the end of a cascade can be calculated on random networks with given degree distribution $p(k)$ and threshold distribution $F_\Theta(\theta)$ [12]. To obtain the results, a branching process approximation was used, also known as heterogeneous mean-field approximation (HMF) or as local tree approximation (LTA) [20]. This approximation was studied in many subsequent works. It was generalized for directed and undirected weighted networks [22,26], it was shown to be accurate even for clustered networks with small mean intervertex distance [19], and the influence of degree-degree correlations has been investigated [20,21]. According to a general framework introduced by Lorenz *et al.* [27], the ED and DD approach belong to the constant load class, where the ED is called the inward variant, while the DD is identified as the outward variant. Still, the risk reduction potential of the latter has not been understood so far, since a system's exposure to systemic risk has been only explored on fully connected or regular networks [27], where both model variants coincide.

In order to study the DD approach on more general networks, we generalized and extended the existing approximations, which were proven to be exact for the case of ED [22,26]. Now, we can treat more general processes where the directed weights in a directed or undirected network can depend on properties of both nodes, the failing as well as the loss facing one. Here, in contrast to Ref. [22], nodes can depend on each other and properties of the failing node influence the amount of loss faced, and in contrast to Ref. [26] nodes can depend on each other in a nonsymmetric way. It is important to note that our approach is still limited to cascade processes where the exposures w_{ji} remain constant over the course of a cascade. But if, for instance, accumulated load [28–30] or overload [31,32] of a node is spread to functional network neighbors, the order of failures and time of failure matter for the cascade outcome. However, the arbitrary choice of weight distributions introduce a significant modeling flexibility.

We show in Sec. IV that our approach leads to very good agreements with simulations on finite Poisson random graphs and scale-free networks. Many large systems belong to the latter class [33–35]. But often, simulations would require more computational time than our analytic approach or would even be impossible, because nodes with degree in the far right tail of the degree distribution are only realized in large networks that most times cannot be sampled with adequate accuracy. Interestingly, right-skewed degree distributions seem to reduce systemic risk for most parameters. An example for such a case is provided in the Appendix.

III. ANALYTIC FRAMEWORK

A. Local tree approximation

In the configuration model a node i is characterized only by its degree k_i so that its failure probability $\mathbb{P}(s_i = 1|k_i)$ depends solely on this information. Hence, the (average) final fraction of failed nodes is of the form

$$\rho = \sum_k p(k)\mathbb{P}(s = 1|k), \tag{4}$$

where $\mathbb{P}(s = 1|k)$ denotes the failure probability of a node conditional on the information that its degree is k.

The quantity ρ allows for two different interpretations. On the system level, ρ measures the final fraction of failed nodes, and $\mathbb{P}(s = 1|k)$ the fraction of failed nodes with degree k. On the node level, ρ can be seen as probability for a node to be failed, but if the node's degree k is known, then its actual failure probability is given by $\mathbb{P}(s = 1|k)$. In the following, we proceed with successively decomposing $\mathbb{P}(s = 1|k)$ into sums over products between conditional probabilities that assume more information about the network neighborhood, and probabilities that the neighborhood is in the assumed state.

1. The conditional failure probability

The computation of $\mathbb{P}(s = 1|k)$ relies on the assumption of an infinite network size ($N \to \infty$), since the clustering coefficient for the configuration model vanishes in the limit, if the second moment of the degree distribution is finite [36]. Consequently, the topology simplifies to locally treelike networks [20], where neighbors of a node are not connected among each other, as illustrated in Fig. 1. In this illustration, the node under consideration with degree k is colored in green and is called the focal node. Its failure probability $\mathbb{P}(s = 1|k)$ decomposes into a sum over two factors. The locally treelike network structure and the assumption that the local neighborhood defines the state of a node already lead to:

$$\mathbb{P}(s = 1|k) = \sum_{n=0}^{k} \mathbb{P}(s = 1|k,n)b(n,k,\pi). \tag{5}$$

The factor $b(n,k,\pi)$ describes the general state of the neighborhood, namely the probability that among the k neighbors of a node exactly n have failed. The factor $\mathbb{P}(s = 1|k,n)$ gives the probability that a node with degree k fails after n of its neighbors have failed. Therefore, it takes into account the ability of a node to withstand shocks (i.e., failing neighbors).

If some of the neighbors of a node would be connected, which violates the local tree-like assumption, we would need to consider all possible (temporal) orders of their failures. Instead, the configuration model allows for assigning every neighbor the same failure probability π and each neighbor's failure can be regarded as independent of the failure of the others because of the locally treelike network structure.

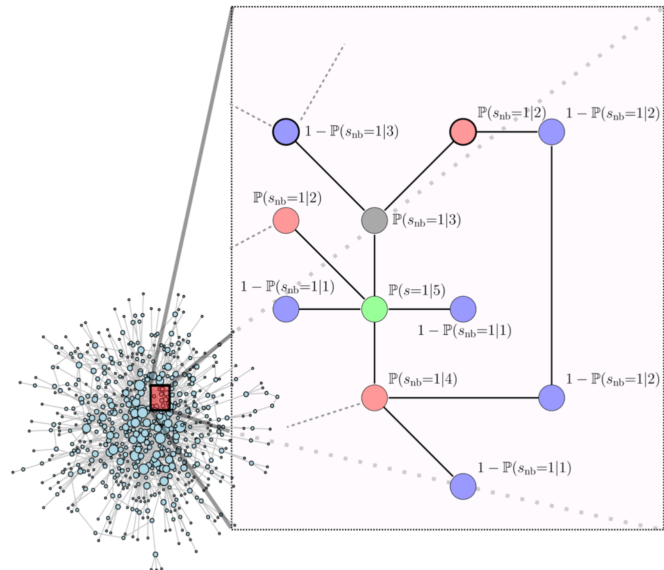

FIG. 1. Illustration of the local tree approximation. The green node is the focal node. Its conditional failure probability $\mathbb{P}(s = 1|k = 5)$ can be computed according to Eq. (5), and depends on the state of its neighbors: Here, the two red ones and the gray one have failed, while the two blue ones are still functional. The neighbors' conditional failure probabilities $\mathbb{P}(s_{\text{nb}} = 1|k)$ rely on the failure probabilities of their own neighbors without regarding the green focal point.

Consequently, the number of failed neighbors of a node is binomially distributed so that n neighbors can fail with probability

$$b(n,k,\pi) = \binom{k}{n}\pi^n(1-\pi)^{k-n}.$$

However, the probability $\mathbb{P}(s = 1|k,n)$ that such an event causes the failure of the considered node with degree k may depend on specific properties of the neighbors, like, e.g., their degrees k_i and their failure probabilities $\mathbb{P}(s_{\text{nb}} = 1|k_i)$. By allowing this dependence, we introduce a generalization of the existing heterogeneous mean-field approximation, which enables the analytical treatment of processes where failing nodes have different influences on their neighbors according to their degree.

Also the conditional failure probability $\mathbb{P}(s = 1|k,n)$ can be decomposed into the sum over two factors,

$$\mathbb{P}(s = 1|k,n) = \sum_{\boldsymbol{k_n} \in I^n} p(\boldsymbol{k_n}|k,n)\mathbb{P}(s = 1|k,\boldsymbol{k_n}), \tag{6}$$

where the sum runs over all possible configurations of neighbors' degrees denoted by I^n. $\boldsymbol{k_n}$ is an abbreviation for a vector $(k_1,\ldots,k_n)$ of failed neighbors' degrees and takes values in I^n. Generally, $I = \mathbb{N}$ or $I = [c] := \{1,\ldots,c\}$ in the presence of a finite cutoff c. Such a cutoff is inevitable in numerical computations or in the observation of (finite) real-world systems, while it guarantees the finiteness of the second moment of $p(k)$.

The probability $p(\boldsymbol{k_n}|k,n)$ captures the precise state of the neighborhood given that exactly n neighbors have failed. More precisely, it is the probability that the n failed neighbors of a node with degree k have degrees $\boldsymbol{k_n}$. This factor may depend on the failing probabilities of neighbors $\mathbb{P}(s_{\text{nb}} = 1|k_1),\dots,\mathbb{P}(s_{\text{nb}} = 1|k_n)$, while conditioned on these failures, we denote $\mathbb{P}(s = 1|k,\boldsymbol{k_n})$ the probability that a node with degree k fails. The latter is determined by the specific cascading model and will be discussed later for the cases of our two diversification models.

2. *The state of the neighborhood*

In order to compute the failure probability,

$$\mathbb{P}(s = 1|k) = \sum_{n=0}^{k} b(n,k,\pi) \sum_{\boldsymbol{k_n}\in I^n} p(\boldsymbol{k_n}|k,n)\mathbb{P}(F|k,\boldsymbol{k_n}), \quad (7)$$

we need to derive the state of the neighborhood as described by the average failure probability of a neighbor π, and the failure probability $p(\boldsymbol{k_n}|k,n)$ that the n failed neighbors have degrees $\boldsymbol{k_n}$. Both depend on the degree distribution of a neighbor $p_n(k)$ as well as its failure probability $\mathbb{P}(s_{\text{nb}} = 1|k)$.

It is important to note that a neighbor with degree k (illustrated by the gray node in Fig. 1) does not fail with probability $\mathbb{P}(s = 1|k)$, since one of its links leads to the focal node (illustrated by the green node in Fig. 1). Conditional on the event that the focal node has not failed yet, only the remaining $k-1$ neighbors of the gray neighbor (colored with bold fringe) could have caused the failure of the gray neighbor of the focal node. This property of a neighbor whose failure does not regard the focal node is called without regarding property (WOR) by Hurd and Gleeson [26]. Therefore, a neighbor's failure probability $\mathbb{P}(s_{\text{nb}} = 1|k)$ is

$$\mathbb{P}(s_{\text{nb}} = 1|k) = \sum_{n=0}^{k-1} b(n,k-1,\pi) \times \sum_{\boldsymbol{k_n}\in I^n} p(\boldsymbol{k_n}|k,n)\mathbb{P}(s = 1|k,\boldsymbol{k_n}). \quad (8)$$

The notation $\mathbb{P}(s_{\text{nb}} = 1|k)$ is a short-cut for conditioning on the event that one neighbor of a node with degree k has not failed.

It remains to calculate $p(\boldsymbol{k_n}|k,n)$ and the unconditional failure probability of a neighbor π. Both depend on the degree distribution $p_n(k)$ of a neighbor. Because of the local tree approximation, it is independent of the degree distribution of the other nodes in the network,

$$p_n(k) := \frac{kp(k)}{z}, \quad (9)$$

where $z := \sum_k kp(k)$ denotes the normalizing average degree. $p_n(k)$ is proportional to the degree k in the configuration model, because each of a neighbor's k links could possibly connect the neighbor with the focal node (see, e.g., Ref. [36]).

We, therefore, obtain the unconditional failure probability π of a neighbor by

$$\pi = \sum_k p_n(k)\mathbb{P}(s_{\text{nb}} = 1|k) = \sum_k \frac{kp(k)}{z}\mathbb{P}(s_{\text{nb}} = 1|k). \quad (10)$$

Similarly, the degree distribution of a neighbor conditional on its failure can be written as $\mathbb{P}(s_{\text{nb}} = 1|k)p(k)k/z\pi$ so that we can calculate the probability $p(\boldsymbol{k_n}|k,n)$ that the n failed neighbors have degrees $\mathbf{k_n}$ by

$$p(\boldsymbol{k_n}|k,n) = \prod_{j=1}^{n} \frac{p(k_j)k_j\mathbb{P}(s_{\text{nb}} = 1|k_j)}{z\pi}, \quad (11)$$

since the neighbors are independent of each other, according to the locally treelike network structure.

3. *Fixed-point iteration for the conditional failure probability*

In short, the vector

$$\mathbb{P}(\mathbf{s_{nb}} = \mathbf{1}|\mathbf{k}) = (\mathbb{P}(s_{\text{nb}} = 1|k))_{k\in\{1,\cdots,c\}} \in [0,1]^c$$

turns out to be a fixed point of a vector valued function $\mathbf{G} : [\mathbf{0},\mathbf{1}]^{\mathbf{c}} \to [\mathbf{0},\mathbf{1}]^{\mathbf{c}}$ so that for the kth component we have

$$\begin{aligned}\mathbb{P}(s_{\text{nb}} = 1|k) &= \sum_{n=0}^{k-1} b(n,k-1,\pi) \sum_{\boldsymbol{k_n}\in I^n} \mathbb{P}(s = 1|k,\boldsymbol{k_n}) \\ &\quad\times \prod_{j=1}^{n} \frac{p(k_j)k_j\mathbb{P}(s_{\text{nb}} = 1|k_j)}{z\pi} \\ &= G_k[\mathbb{P}(\mathbf{s_{nb}} = \mathbf{1}|\mathbf{k})], \end{aligned} \quad (12)$$

where $\mathbb{P}(s_{\text{nb}})$ denotes the failure probability of a neighbor with degree k, $b(n,k-1,\pi) = \binom{n}{k}\pi^n(1-\pi)^{k-1-n}$ belongs to a binomial distribution with respect to the failure probability π of a neighbor, $p(k)$ is the degree distribution with mean z, and $\mathbb{P}(s = 1|k,\boldsymbol{k_n})$ stands for the probability of a node to fail if exactly n of its k neighbors with degrees $k_1,\dots,k_n$ have failed before.

Such a fixed point exists according to the Knaster-Tarski theorem, since the function $\mathbf{G}$ is monotone with respect to a partial ordering and maps the complete lattice $[0,1]^c$ onto itself [37]. A proof is given in Appendix A.

Thus, starting from an initial vector $\mathbb{P}(\mathbf{s_{nb}} = \mathbf{1}|\mathbf{k})^{(\mathbf{0})}$, which is defined by the considered cascading model, we can compute the fixed point iteratively by

$$\mathbb{P}(\boldsymbol{s_{nb}} = \mathbf{1}|\boldsymbol{k})^{(t+1)} = \boldsymbol{G}(\mathbb{P}(\boldsymbol{s_{nb}} = \mathbf{1}|\boldsymbol{k})^{(t)}), \quad (13)$$

with

$$\pi^{(t)} = \sum_k p_n(k)\mathbb{P}(s_{\text{nb}} = 1|k)^{(t)}. \quad (14)$$

Each iteration step (t) corresponds to one discrete time step of the cascading process so that

$$\rho^{(t)} = \sum_k p(k)\mathbb{P}(s = 1|k)^{(t)} \quad (15)$$

can be interpreted as average fraction of failed nodes in the network at time t. Note that the relation between $\mathbb{P}(\boldsymbol{s} = \mathbf{1}|\boldsymbol{k})^{(t)}$ and $\mathbb{P}(\boldsymbol{s_{nb}} = \mathbf{1}|\boldsymbol{k})^{(t)}$ is described by Eqs. (7) and (11).

4. *Simplification for homogeneous failure probability*

In case the impact of a failing neighbor does not depend on its degree, the failure probability $\mathbb{P}(s = 1|k,\boldsymbol{k_n}) = \mathbb{P}(s = 1|k,n)$ does not depend on the degrees $\boldsymbol{k_n}$ of its n failed neighbors, and Eq. (12) can be simplified to

$$\mathbb{P}(s_{\text{nb}}=1|k)=\sum_{n=0}^{k-1}b(n,k-1,\pi)\mathbb{P}(s=1|k,n)\times\prod_{j=1}^{n}\frac{1}{\pi}\sum_{k_j\in I}\frac{p(k_j)k_j\mathbb{P}(s_{\text{nb}}=1|k_j)}{z}=\sum_{n=0}^{k-1}b(n,k-1,\pi)\mathbb{P}(s=1|k,n), \tag{16}$$

using Eq. (10). Inserting this into Eq. (10) leads to the fixed-point equation

$$\pi=\sum_k\frac{kp(k)}{z}\sum_{n=0}^{k-1}b(n,k-1,\pi)\mathbb{P}(s_{\text{nb}}=1|k,n), \tag{17}$$

which in this case involves the scalar π instead of the vector $\boldsymbol{\mathbb{P}(s_{\text{nb}}=1|k)}$ in Eq. (12). With this information the final fraction of failed nodes can be computed as

$$\rho=\sum_k p(k)\sum_{n=0}^{k}b(n,k,\pi)\mathbb{P}(s=1|k,n), \tag{18}$$

as already known from the literature [12,20]. Still, this simpler approach is not able to capture the cascade dynamics of the damage diversification model.

5. The ability of a node to withstand a shock

The only piece missing in our derivations is the model-specific probability $\mathbb{P}(s=1|k,\boldsymbol{k_n})$, that a node with degree k fails exactly after n of its neighbors with degrees $\boldsymbol{k_n}$ have failed. This probability captures the failure dynamics and is thus defined by a node's total loss $L(k,\boldsymbol{k_n})$ and its threshold $\Theta(k)$. Here, we write $\Theta(k)$ to indicate that it is a random variable that can depend on the degree k of a node. But later on, in the Results section, we only test for cases where Θ is independent from k. Each node i receives a threshold θ_i randomly drawn initially according to the law $F_{\Theta(k)}$ of the random variable $\Theta(k)$, but θ_i itself stays constant for the whole cascading process.

The given information about the degrees $k,\boldsymbol{k_n}$ can in principle enter both variables, $L(k,\boldsymbol{k_n})$ and $\Theta(k)$, although the cumulative threshold distribution $F_{\Theta(k)}$ tends to depend solely on properties of the node itself, e.g., the degree k. Because a node fails, if its total loss exceeds its threshold, we have

$$\mathbb{P}(s=1|k,\boldsymbol{k_n})=\mathbb{P}[\Theta(k)\leqslant L(k,\boldsymbol{k_n})]. \tag{19}$$

Please note that initially the total loss of a node is zero, since no neighbors have failed because of neighboring failures yet, i.e., $n=0$. Consequently, all nodes with threshold $\theta_i\leqslant 0$ fail in the beginning and trigger a cascade. In an infinitely large network, these are $F_{\Theta(k)}(0)$ of all nodes with degree k.

More generally, with respect to known weight distributions $p_{W(k_j,k)}$ between a neighbor with given degree k_j and a node with degree k (and thus a weight of a link starting in a neighbor with degree k_j and ending in a node with degree k), Eq. (19) reads

$$\mathbb{P}(s=1|k,\boldsymbol{k_n})=\mathbb{P}\left(\Theta(k)\leqslant\sum_{j=1}^{n}W(k_j,k)\right)=\int F_{\Theta(k)}(w)(p_{W(k_1,k)}*\ldots*p_{W(k_n,k)})(w)\,dw\,. \tag{20}$$

The last equation holds if the weight distributions $p_{W(k_j,k)}$ are independent, and the symbol $*$ denotes a convolution.

In the simpler case of our two model variants, the weights $W(k_j,k)$ are completely deterministic. In accordance with the definition of the weights in Eq. (3), we calculate for the ED case

$$\mathbb{P}(s=1|k,\boldsymbol{k_n})^{(\text{ED})}=\mathbb{P}(s=1|k,n)^{(\text{ED})}=F_{\Theta(k)}\left(\frac{n}{k}\right), \tag{21}$$

which is independent of the neighbors' degrees. Consequently, the calculation of the average final fraction of failed nodes can be simplified as outlined in Sec. III A 4.

For the DD case, the fixed point iteration needs to take into account all degrees of the failed neighbors, since they define the loss $1/k_j$ that the focal node faces. Thus, we have

$$\mathbb{P}(s=1|k,\boldsymbol{k_n})^{(\text{DD})}=F_{\Theta(k)}\left(\sum_{j=1}^{n}\frac{1}{k_j}\right). \tag{22}$$

6. DD case: Correct Heterogeneous mean-field approximation (cHMF)

The probability of the failure of a node or neighbor with degree k and n failed neighbors $\mathbb{P}(s=1|k,n)^{(\text{DD})}$ given by Eq. (6) needs to be recalculated for each fixed point iteration step. For the DD case, this involves the calculation of the convolution of the impact distribution (or inflicted loss distribution) p_{imp} of a failed neighbor, which depends on the iteratively updated failure probability $\mathbb{P}(s_{\text{nb}}=1|k)$. More precisely, one failed neighbor inflicts the loss $1/k$ with probability

$$p_{\text{imp}}\left(\frac{1}{k}\right)=\mathbb{P}(s_{\text{nb}}=1|k)\frac{kp(k)}{z\pi} \tag{23}$$

(that is conditioned on its failure) independently of the other failed neighbors. Thus, the total loss $L(k,n)$ of a node with degree k and n failed neighbors is distributed according to the nth convolution of this impact distribution p_{imp}^{*n} and we have

$$\mathbb{P}(s=1|k,n)=\mathbb{P}(\Theta(k)\leqslant L(k,n))=\sum_l p_{\text{imp}}^{*n}(l)F_{\Theta}(l),$$

where the inner sum runs over all possible values of the total loss L.

Since the convolutions are computationally demanding (in terms of time and especially memory), we approximate p_{imp}^{*n} by first binning it to an equidistant grid and then using fast Fourier transformations (FFT) in order to take advantage of the fact that convolutions correspond to simple multiplications in Fourier space (see, for instance, Refs. [38,39]). This reduces the computation of the fraction of failed nodes for fixed threshold and degree distribution parameters in our setup to

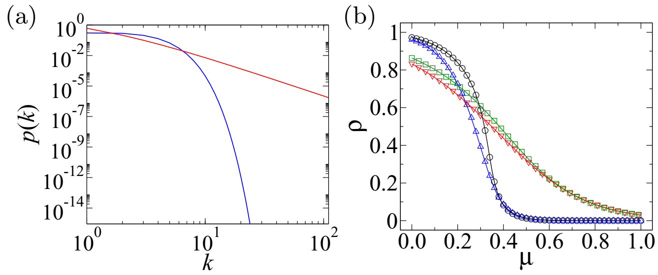

FIG. 2. (a) The studied degree distributions: Poisson distribution with parameter $\lambda = 2.82$ and cutoff degree $c = 50$ (blue), scale-free distribution with exponent $\gamma = 3$ and maximal degree $c = 200$ and average degree $z = 3$ (red) in log-log scale. (b) Comparison of the final fraction of failed nodes obtained by simulations (symbols) on networks with 100 000 nodes as average over 500 independent realizations with numerical results from the cHMF (lines) for the DD case. The thresholds Θ are normally distributed with mean μ and standard deviation σ ($\Theta \sim \mathcal{N}(\mu,\sigma^2)$). Nodes with negative thresholds fail initially. Results for scale free networks are depicted in blue for $\sigma = 0.2$ and red for $\sigma = 0.5$. Results for networks with Poisson degree distribution are shown in black for $\sigma = 0.2$ and green for $\sigma = 0.5$.

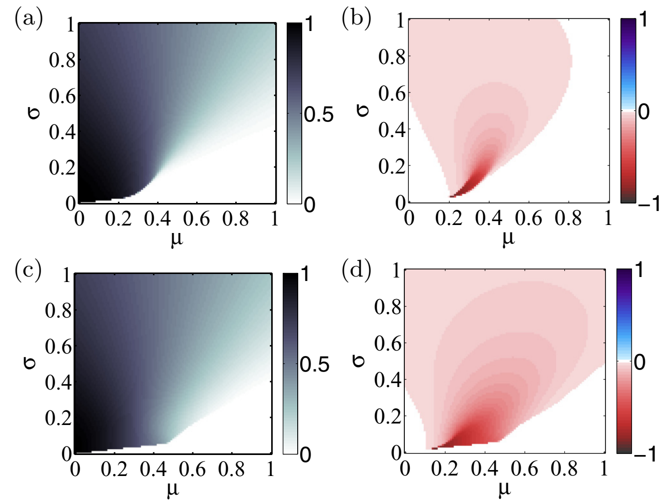

FIG. 3. (a) Fraction of failed nodes obtained by (simpHMF) for the DD case on Poisson random networks with $\lambda = 2.82$, $z = 3$, and $c = 50$. (b) Difference between the correct version (cHMF) and (a). (c) Fraction of failed nodes obtained by (simpHMF) for the DD case on scale free networks with $\gamma = 3$, $z = 3$, and $c = 200$. (d) Difference between the corresponding correct version (cHMF) version and (c). The thresholds Θ are normally distributed with mean μ and standard deviation σ ($\Theta \sim \mathcal{N}(\mu,\sigma^2)$).

a few minutes. Although this is numerically accurate enough for the calculation of the final fraction of failed nodes, for the reporting of the vectors $\mathbb{P}(\mathbf{s} = \mathbf{1}|\mathbf{k})$ and $\mathbb{P}(\mathbf{s}_{\text{nb}} = \mathbf{1}|\mathbf{k})$ we use a more precise direct convolution of the binned impact distributions p_{imp}, as is described in Appendix B. Figure 2(b) shows that our numerical results coincide with simulations.

7. Neglecting the neighbors' degrees in the failure probability (simpHMF)

Considering the computational complexity of the cHMF approach we described above, it is worth asking whether we can approximate it with a simpler version as, e.g., the one described in Sec. III A 4, and still obtain reasonably good results.

This would require the probability $\mathbb{P}(s = 1|k,\boldsymbol{k_n})$ to be independent of the neighbors' degree. Consequently, we would assume that every failed neighbor inflicts the loss $1/k$ with probability

$$p_{\text{simp}}\left(\frac{1}{k}\right) = \frac{kp(k)}{z}, \tag{24}$$

although its degree does not need to coincide with k. Therefore, by computing

$$\mathbb{P}(s = 1|k,\boldsymbol{k_n}) = \sum_{l} p_{\text{simp}}^{*n}(l)F_{\Theta}(l) = \mathbb{P}(s = 1|k,n), \tag{25}$$

we can calculate the failure probability initially without the need to update it in each fixed-point iteration. Although this approach is more convenient, as shown in Fig. 3, it is inadequate for the damage diversification variant, especially in combination with skew degree distributions (as, e.g., in case of scale free networks). This is because if we follow this simplified calculation, we lose the risk-reducing effect by hubs that are connected to more nodes, and we would draw opposite conclusions about systemic risk. So, as shown in Fig. 3, it is crucial to use the correct HMF to explain our simulation results. The simplified method simpHMF leads to an overestimation of the final fraction of failed nodes.

IV. NUMERICAL RESULTS

Hereafter, we show how the DD variant can reduce systemic risk in comparison to the ED variant by calculating the fraction of failed nodes for both variants and several degree distributions. Alongside, we study the influence of the presence of hubs and the degree variance. This allows us to discuss up to which extent the overall diversification reduces or increases systemic risk. Later on, we also compare how a higher or lower diversification influences the failure risk of a node on the mesoscopic level.

Our findings do not only lead to different conclusions for the two model variants, but also depend on the threshold distribution parameters under consideration.

Similar to Refs. [14] and [12], we study normally distributed thresholds $\Theta \sim \mathcal{N}(\mu,\sigma^2)$ with mean μ and standard deviation σ, but we explore the role of the thresholds' heterogeneity as well as their mean size more extensively by providing a 2D phase diagram as in Ref. [27]. Although our analytic framework also applies to more general cases, here we assume the thresholds to be independent from the degree k of a node.

The initial fraction of failed nodes is determined by the nodes with negative thresholds ($\Theta \leqslant 0$) and is thus given by $F_{\Theta}(0) = \Phi(\frac{-\mu}{\sigma})$, where Φ denotes the cumulative distribution function of the standard normal distribution.

A. Systemic perspective

In principle, all phase diagrams are of similar shape as the one for fully connected networks of infinite size that has been calculated in Ref. [27] and is depicted in Fig. 4(a). It shows

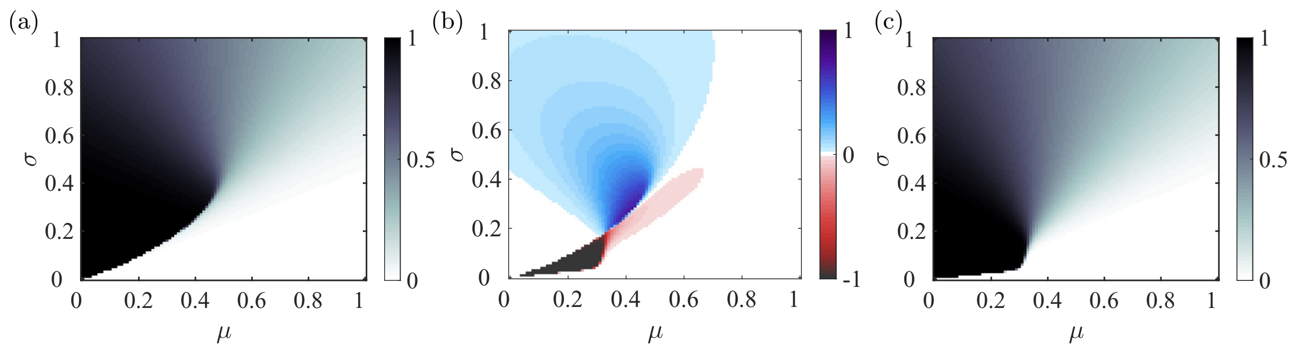

FIG. 4. Phase diagram for the fraction of failed nodes ρ with normally distributed thresholds ($\Theta \sim \mathcal{N}(\mu,\sigma^2)$) for (a) fully connected networks, and (c) regular networks with degree $z = 3$. The darker the color the higher is the systemic risk. The middle panel (b) shows their difference $\rho^{(a)} - \rho^{(b)}$.

the effect of perfect diversification. Both variants, ED and DD, coincide for this case as well as for regular networks where every node has the same degree z.

We observe that increasing the standard deviation can also reduce systemic risk—even though this increases the fraction of initial failures. Of special prominence is the sharp regime shift from a region of small systemic risk to an almost complete system breakdown. The existence of such a regime shift has been first identified for Poisson random graphs and the ED variant in Ref. [12]. It is present for most other topologies as well—for both variants ED and DD. The regime shift separates the white region in Fig. 4(a), where perfect diversification is (nearly) optimal, from a region where other topologies could expose the system to a lower systemic risk. For instance, regular networks with degree z reduce the risk for larger threshold parameters σ, but also increase the risk for small μ and σ. Figure 4(c) shows the respective phase diagram for $z = 3$. Consequently, we cannot expect that other topologies different from the fully connected one lead to smaller systemic risk for all parameters. But the DD variant can expose the system to a lower systemic risk in comparison with ED on the same topology. And both variants can show lower systemic risk for such a topology than fully connected networks for certain threshold parameters.

We test two degree distributions where the ED and DD variants differ. Figure 5 presents the final fraction of failed nodes for Poisson random graphs and scale-free networks with degree distributions

$$p_P(k) := \frac{1}{S_P}\frac{\lambda^k}{k!}, \quad p_S(k) := \frac{1}{S_S}\frac{1}{k^\gamma}$$

for $k \in \{1, \cdots, c\}$ with normalizing constants

$$S_P := \sum_{k=1}^{c} \frac{\lambda^k}{k!} \quad \text{and} \quad S_S := \sum_{k=1}^{c} \frac{1}{k^\gamma},$$

where we adjust $p_S(1)$ (and S_S) to set the average degree z of p_S to a specific value in the aftermath. The Poisson random graphs are of interest as limit of the well-studied Erdös-Rényi random graphs [40], and in comparison to the simulations serve as benchmark for our method [see Fig. 2(b)]. However, many real-world systems are expected to be of scale-free nature [33,41,42]. The scale-free degree distribution is especially interesting for our diversification analysis, since a considerable fraction of nodes has a higher degree than others and can therefore be called hubs. These hubs are the well-diversified nodes.

In Fig. 5 we compare the two variants ED and DD for the two degree distributions with the same average degree $z = 3$ so that we can study the influence of the presence of hubs rather than the overall connectivity indicated by z. We observe that the outcome for the ED variant differs for the two degree distributions only in a narrow threshold parameter range. While the DD variant exposes the system to a smaller systemic risk than the ED variant for all parameters, it especially proves risk reducing for higher degree variation. Scale-free random graphs expose the system to lower systemic risk than Poisson random graphs, and Poisson random graphs have lower systemic risk than regular random graphs. The presence of hubs, whose failure causes only small damage in their environment, seems to limit the cascade risk. But in general, a majority of hubs does not reduce systemic risk. Cascades in fully connected networks, where every node is a hub, are more amplified for already identified threshold parameters than in other studied topologies. The DD variant together with a high diversity of degrees, so that a small fraction of high degree nodes is combined with a majority of small degree nodes whose failure does not affect a high proportion of other nodes, lowers systemic risk considerably in threshold parameter regions where the system is vulnerable to failure cascades.

We present further examples in the Appendix. In fact, a small set of threshold parameters can be found where the ED variant exposes the system to lower systemic risk than DD. This can also be observed for a degree distribution measured from a snapshot of the Italian interbank market in October 1, 2002, as published in Ref. [43]. But the DD variant shows lower systemic risk than ED for most parameters also in this case. Additionally, we provide an example where the sudden regime shift in the phase diagram vanishes. Scale free random graphs with very low average degree z show in general only small systemic risk (that is even smaller for DD than ED).

The presented results demonstrate that not only the overall connectivity z and robustness (represented by the threshold parameters) determine the final cascade size. Other topological system properties as the specific distributions of degrees and link weights w_{ij} have risk reduction potential as well.

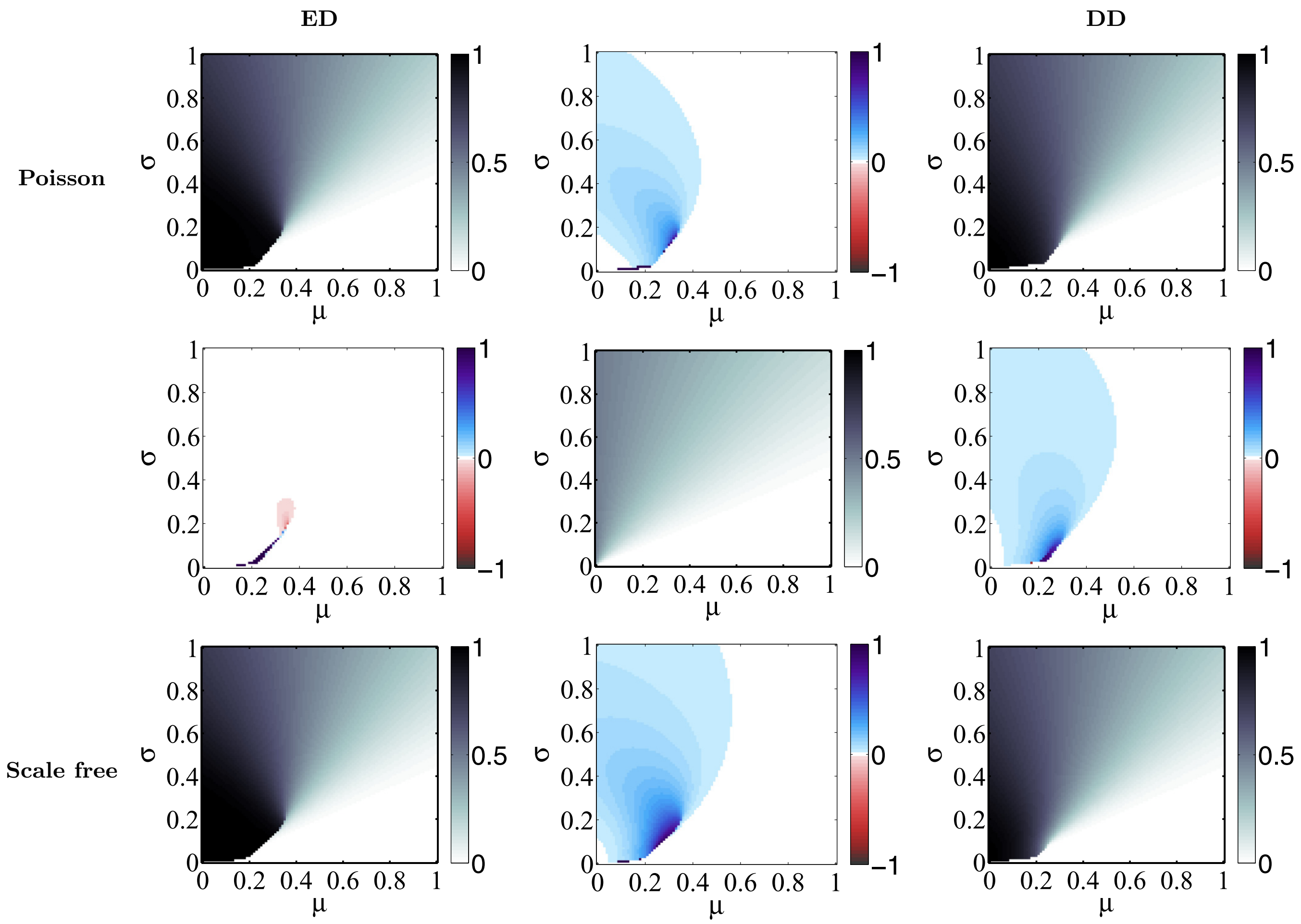

FIG. 5. Phase diagrams for the final fraction of failed nodes ρ calculated numerically for different degree distributions with average degree $z = 3$, diversification variants ED and DD, and their differences. The thresholds Θ are normally distributed with mean μ and standard deviation σ ($\Theta \sim \mathcal{N}(\mu,\sigma^2)$). The fraction of initially failed nodes is given by $F_\Theta(0)$. The darker the color the higher is the systemic risk. First row: Poisson distribution with parameter $\lambda = 2.82$, and cutoff degree $c = 50$ for the ED (left) and DD (right). The middle panel shows their difference $\rho^{(\mathrm{ED})} - \rho^{(\mathrm{DD})}$. Second row: The difference between the diagrams with Poisson and scale-free degree distributions for the ED variant (left). Similarly for the DD variant (right). In the middle panel the initial fraction of failed nodes $\rho(0)$ is illustrated. $\rho_o := \rho(0)$ is constant along the lines $\sigma = \mu/\Phi^{-1}(\rho_0)$. Third row: Scale-free distribution with exponent $\gamma = 3$, and maximal degree $c = 200$ for the ED (left) and DD (right). The middle panel again shows their difference $\rho^{(\mathrm{ED})} - \rho^{(\mathrm{DD})}$.

The interplay of hubs and leaves seems to have an important effect. To understand their different roles in the cascade amplification, we study next the failure risk of nodes conditional on their degree.

B. Mesoscopic perspective

Two quantities reveal the role of the nodes with a given degree k in a cascade process: (a) their failure probability $\mathbb{P}(s = 1|k)$ and (b) the cascade amplification that their failure triggers. We measure the second quantity (b) as the increase of the fraction of failed nodes ρ as response to an increase of a neighbors' failure probability $\mathbb{P}(s_{\mathrm{nb}} = 1|k)$. Thus, we call the partial derivative,

$$\frac{\partial \rho}{\partial \mathbb{P}(s_{\mathrm{nb}} = 1|k)},$$

cascade amplification by a node with degree k.

For the ED variant the conditional failure probability (a) does not depend explicitly on the degree distribution and thus, on the diversification strategies of the other nodes. It is determined only by the failure probability of a neighbor π, which indicates the state of the system in the cascade, and the threshold distribution. Equations (17) and (21) give

$$\mathbb{P}(s = 1|k) = \sum_{n=0}^{k} b(n,k,\pi) F_\Theta\left(\frac{n}{k}\right).$$

We would expect that large degree, and thus, high diversification decreases the failure risk, since the failure of a high number of neighbors is less probable than the failure of fewer neighbors. As shown in Fig. 6(a), this intuition applies only to cases of small risk where the failure probability π of a neighbor is small. Here, π is chosen independently of a cascade evolution. For a few parameters with small threshold standard deviation σ, also more irregular shapes of the conditional failure probability are possible (see, e.g., the red circles in

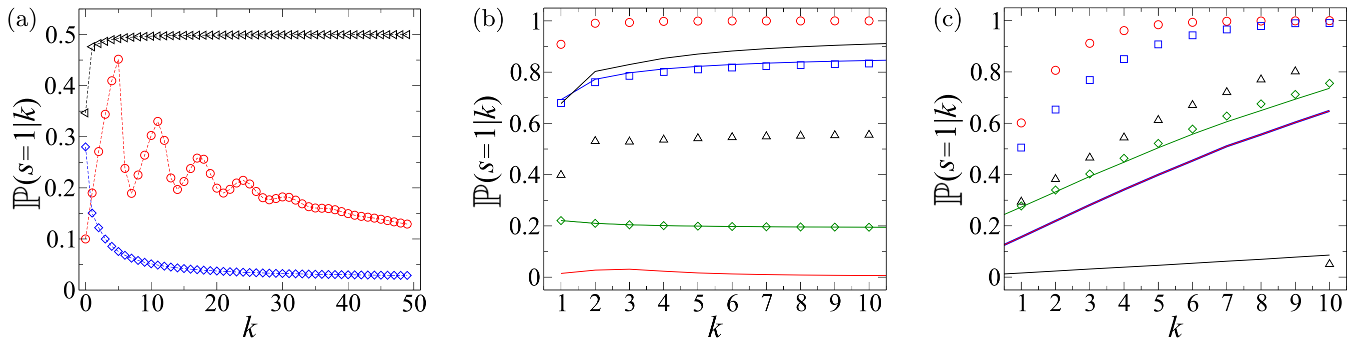

FIG. 6. (a) Conditional failure probability for ED and normally distributed thresholds with parameters $(\mu,\sigma) = (0.3,0.2)$ with $\pi = 0.2$ (black triangles), $(\mu,\sigma) = (0.15,0.01)$ with $\pi = 0.1$ (red circles), and $(\mu,\sigma) = (0.7,0.2)$ with $\pi = 0.3$ (blue diamonds). Here, π is chosen independent of a cascade process. (b) Conditional failure probability for ED and normally distributed thresholds with parameters $(\mu,\sigma) = (0.27,0.09)$ (red), $(\mu,\sigma) = (0.3,0.3)$ (blue), $(\mu,\sigma) = (0.34,0.19)$ (black), $(\mu,\sigma) = (0.7,0.6)$ (green), and π defined by cascade equilibrium. Lines indicate a scale free distribution with $\gamma = 3$, $c = 200$, and $z = 3$, while symbols indicate Poisson random graphs with $\lambda = 2.82$, $z = 3$, and $c = 50$. (c) As in (b), but for DD while in this case green color refers to parameters $(\mu,\sigma) = (0.5,0.5)$. Nodes with degree $k > 10$ have a similar failure probability as a node with degree $k = 10$.

Fig. 6(a)). Still, in a real cascade process, the cascade is ongoing in those cases until most nodes are failed. Results of real ED cascade processes are depicted in Fig. 6(b), where π is calculated in a fixed point iteration corresponding to the studied threshold distribution. In this case, hubs have a higher risk to fail in a cascade than leaves for almost all threshold parameters. Exceptions occur, e.g., $(\mu,\sigma) = (0.7,0.6)$, but then the difference in failure risk between hubs and leaves is negligible. Initially, hubs often have a smaller failure risk than leaves. But if cascades get amplified and π increases, hubs face a larger failure risk than leaves. High diversification becomes unfavorable. Intuitively, nodes with a higher degree are more exposed to an ongoing cascade, as they have a higher chance to be hit by it. But the diversification effect most often saturates for degrees $k > 10$. Because of this we restrict Fig. 6(a) to $k \leqslant 10$. The failure probability of a node with degree $k = 10$ differs only marginally from the one of a node with higher degree.

This explains why we observe similar phase diagrams for Poisson and scale-free random graphs for the ED variant. The ED variant responds barely to changes in the tail of a degree distribution, as nodes with large degrees have almost identical failure risk.

In contrast, for the DD variant, the outcome ρ depends crucially on the degree distribution. Still, hubs have a higher failure risk than nodes with a smaller degree in general; see Fig. 6(c). Each additional neighbor introduces the possibility of a loss, if it fails. Thus, the shape of the conditional failure probability,

$$\mathbb{P}(s = 1|k) = \sum_{n=0}^{k} b(n,k,\pi) \sum_{l} p_{\mathrm{imp}}^{*n}(l) F_\Theta(l),$$

always looks similar as the ones presented in Fig. 6(c). Consequently, too many nodes with high degrees would increase the vulnerability of the system.

However, the presence of a few hubs also decreases the overall failure risk by decreasing some of the possible losses. We gain this insight by studying the second indicator (b) that describes the role of nodes in the cascade amplification. For both variants, ED and DD, the partial derivative of ρ is of the form

$$\frac{\partial\rho}{\partial\mathbb{P}(s_{\mathrm{nb}} = 1|k)} = p(k)k[C_{\mathrm{ED/DD}} - \pi]. \tag{26}$$

The constant $C_{\mathrm{ED/DD}}$ can be interpreted as failure probability of a neighbor where one of its neighbors is failed and has degree k. Thus, the infinitesimal change in ρ is in fact proportional to the increase of the failure probability of a neighbor.

In the case of the ED variant, the constant C_{ED} does not depend on the degree k of a node whose failure probability has increased:

$$C_{\mathrm{ED}} = \sum_{d} \frac{p(d)d}{z} \sum_{n=0}^{d-1} b(n,d-1,\pi) F_\Theta\left(\frac{n+1}{d}\right).$$

Consequently, the failure of hubs is especially problematic, as the cascade amplification is proportional to the degree k of a failed node. The only way to reduce systemic risk (while preserving the average degree z) in comparison to regular random graphs is to introduce nodes to the system with smaller degree than z. Then, only a small fraction of hubs needs to exist in order to preserve the average degree z. But the risk reduction is obtained by the high number of small degree nodes.

In the case of the DD variant, risk reduction is also obtained by the existence of hubs. The constant C_{DD} decreases with degree k:

$$C_{\mathrm{DD}} = \sum_{d} \frac{p(d)d}{z} \sum_{n=0}^{d-1} b(n,d-1,\pi) \sum_{l} p_{\mathrm{imp}}^{*n}(l) F_\Theta\left(l + \frac{1}{k}\right).$$

The cascade amplification [Eq. (26)] role of a node is decided by the tradeoff between the increasing factor $p(k)k$ and the decreasing factor $C_{\mathrm{DD}} - \pi$. Especially less diversified nodes, which have a chance to survive also large failure cascades, can benefit from the diversification of others.

V. DISCUSSION

Although risk diversification is generally considered to lower the risk of an individual node, on the system level it can even lead to the amplification of failures. With our work, we have deepened the understanding of cascading failure processes in several ways.

At first, we generalize the method to calculate the systemic risk measure, i.e., the average cascade size, to include directed and weighted interactions. The loss that a node experiences because of the failure of a neighbor can depend now also on properties of the neighbor and not only on properties of the node itself.

The final cascade size as macro measure is complemented by a measure on the meso level, by calculating individual failure probabilities of nodes based on their degree (diversification).

This allows us to compare two different diversification mechanisms: ED (exposure diversification) and DD (damage diversification). As we demonstrate, nodes that diversify their exposures well (i.e., hubs), have a lower failure risk only as long as the system as a whole is relatively robust. But above a certain threshold for the failure probability of neighboring nodes, such hubs are at higher risk than other nodes because they are more exposed to cascading failures. This effect tends to saturate for large degrees.

In general, most regulatory efforts follow the *too big to fail* strategy and focus on the prevention of the failures of systemic relevant nodes—the hubs. This is mainly achieved by an increase of the thresholds, i.e., capital buffers in a financial context or immunization in case of epidemic spreading. But in reality, this is often very costly.

With our study of another diversification strategy, the damage diversification, we suggest to accompany regulatory efforts by mitigating the failure of hubs. By limiting the loss that every node can impose on others, the damage potential of hubs and, thus, the overall systemic risk is significantly reduced. While this is systemically preferable, the DD strategy is a two-edged sword: Hubs face an increased failure risk, but many small degree nodes benefit from the diversification of their neighbors. This lowers the incentives for diversification as long as no other benefit, e.g., higher gains in times of normal system operation, comes along with a high degree.

As we show, the systemic relevance of a node is not solely defined by its degree, or connectivity. The size of its impact in case of its failure, and thus its ability to cause further failures, is crucial. It is a strength of our approach that we can calculate this impact analytically and obtain a more refined and realistic identification of system-relevant nodes. A possible indicator for system relevance is the cascade amplification measure that we have derived.

Additionally, our approach can be transferred to degree-degree correlated networks [20,21]. We would expect that a high-degree assortativity in the DD could further reduce systemic risk, since the failure risk of diversified nodes could be reduced by connections to hubs whose failures would impact their neighborhood only little.

With our work we have given one example for systemic risk reduction by topological means. Further possibilities can be explored with the analytical framework that we have provided.

ACKNOWLEDGMENTS

R.B. acknowledges support by the ETH48 project; R.B., A.G., and F.S. acknowledge financial support from the Project No. CR12I1_127000 *OTC Derivatives and Systemic Risk in Financial Networks* financed by the Swiss National Science Foundation. A.G. and F.S. acknowledge support by the EU-FET, Project No. MULTIPLEX 317532.

APPENDIX A: EXISTENCE OF A FIXED POINT

We are going to prove with the help of the Knaster-Tarski Theorem that the function $\mathbf{G}$ in Eq. (12),

$$\begin{aligned}\mathbb{P}(s_{\rm nb}=1|k) &= \sum_{n=0}^{k-1} b(n,k-1,\pi) \sum_{\boldsymbol{k_n}\in I^n} \mathbb{P}(s=1|k,\boldsymbol{k_n}) \\ &\quad \times \prod_{j=1}^{n} \frac{p(k_j)k_j \mathbb{P}(s_{\rm nb}=1|k_j)}{z\pi} \\ &= G_k[\mathbb{P}(\mathbf{s}_{\rm nb}=\mathbf{1}|\mathbf{k})],\end{aligned}$$

attains a fixed point. We need to show that $\mathbf{G}$ is monotone with respect to a partial ordering and maps the complete lattice $[0,1]^c$ onto itself.

The partial ordering on $[0,1]^c$ is defined by the following: For any two vectors $\mathbf{x}, \mathbf{y} \in [\mathbf{0},\mathbf{1}]^{\mathbf{c}}$ holds $\mathbf{x} \leqslant \mathbf{y}$, if and only if $x_i \leqslant y_i$ holds for all their components $i \in I$.

First, we note that for the zero vector $\mathbf{0} \in [0,1]^c$ (with 0 in each component) is smallest vector in $[0,1]^c$ with respect to this ordering, while $\mathbf{1} \in [0,1]^c$ (with 1 in each component) is the biggest one. (And both vectors can be compared with each other vector in $[0,1]^c$.)

Consequently, the properties $\mathbf{G}(\mathbf{0}),\mathbf{G}(\mathbf{1}) \in [0,1]^c$ and $\mathbf{G}$ monotone imply that $\mathbf{G}$ is also onto, i.e., $\mathbf{G}([0,1]^c) \subseteq [0,1]^c$.

We can see immediately that $\mathbf{G}(\mathbf{0}) = (F_\Theta(0))_{k=1,\ldots,c} \in [0,1]^c$. For $\mathbf{1} \in [0,1]^c$ and thus $\mathbb{P}(s_{\rm nb}=1|k)=1$ we have

$$\pi = \sum_k \mathbb{P}(s_{\rm nb}=1|k)kp(k)/z = \sum_k 1\cdot kp(k)/z = 1,$$

so that we obtain for the dth component of $\mathbf{G}(\mathbf{1})$,

$$G_d(\mathbf{1}) = \sum_{\mathbf{k_{d-1}}\in I^{d-1}} \mathbb{P}(s=1|d,\mathbf{k_{d-1}}) \prod_{j=1}^{d-1} \frac{p(k_j)k_j}{z} \in [0,1],$$

since $0 \leqslant \mathbb{P}(s=1|d,\mathbf{k_{d-1}}) \leqslant 1$.

It is left to show that $\mathbf{G}$ is monotone with respect to the introduced partial ordering.

We recall that $\mathbf{G}$ maps a vector $\mathbf{v} \in [0,1]^c$, whose kth entry v_k can be interpreted as conditional failure probability of a neighbor with degree k, to another vector $\mathbf{G}(\mathbf{v})$ that we interpreted as well as conditional failure probabilities, but one iteration further in a cascade.

This allows us to reformulate $G_d(\mathbf{v})$ in a way that simplifies the analysis of its monotonicity. As failure probability of a neighbor with degree d we can express $G_d(\mathbf{v})$ in terms of the inflicted loss distribution by a single neighbor on the focal neighbor under consideration: $p_{\rm il}$. We write $p_{\rm il}$ instead of $p_{\rm imp}$ as for the impact distribution, since we do not condition on the failure of a neighbor as in $p_{\rm imp}$. Thus, if a neighbor has not failed, it simply inflicts the loss 0. This event is considered in the first summand (given the fact that a neighbor with degree

k fails with probability v_k):

$$p_{\text{il}}(0) := \sum_k (1 - v_k)\frac{p(k)k}{z} + \\ + \sum_k v_k \frac{p(k)k}{z}\mathbb{P}(W(k,d) = 0) \\ =(1 - \pi) + \sum_k v_k \frac{p(k)k}{z}\mathbb{P}(W(k,d) = 0).$$

The second sum adds the probability of the event that the neighbor has failed but does not inflict a loss. For $x > 0$ we have

$$p_{\text{il}}(x) := \sum_k v_k \frac{p(k)k}{z}\mathbb{P}(W(k,d) = x),$$

since a neighbor has a degree k with probability $p(k)k/z$, conditional on that is failed with probability v_k and inflicts a loss x with probability $\mathbb{P}(W(k,d) = x)$ to a node with degree d.

If instead of a single neighbor, $d - 1$ neighbors can inflict a loss the total loss (or loss of the neighbor that experiences the losses) is distributed by the $(d - 1)$th convolution of the inflicted loss distribution by a single neighbor: $p_{\text{il}}^{*^{d-1}}$.

Consequently, we can write

$$G_d(\mathbf{v}) = \sum_x p_{\text{il}}^{*^{d-1}}(x) F_{\Theta(d)}(x) \tag{A1}$$

or

$$G_d(\mathbf{v}) = \int_0^\infty p_{\text{il}}^{*^{d-1}}(x) F_{\Theta(d)}(x)\, dx, \tag{A2}$$

if the weights $W(k,d)$ are (absolutely) continuously distributed.

In this form, it is straightforward to see that $\mathbf{G}$ is monotone. Let's assume that we have two vectors $\mathbf{v},\mathbf{w} \in [0,1]^c$ with $\mathbf{v} \leqslant \mathbf{w}$. Without loss of generality we further assume that they only differ in one component j, i.e., $\Delta := w_j - v_j$. If they differ in more than one component, we can argue as we proceed separately for each component.

We denote the inflicted loss distribution with respect to v by $p_{\text{il,v}}$ and for w with $p_{\text{il,w}}$. From their definition we can deduce the relation

$$p_{\text{il,w}}(x) = p_{\text{il,v}}(x) + \Delta \frac{p(j)j}{z}\mathbb{P}(W(j,d) = x)$$

for $x > 0$, while for $x = 0$ we observe with

$$p_{\text{il,w}}(0) = p_{\text{il,v}}(0) - \Delta \frac{p(j)j}{z}(1 - \mathbb{P}(W(j,d) = 0))$$

a decrease in probability mass (if $\Delta > 0$). Thus, the inflicted loss distribution by a single neighbor is shifted for $\mathbf{w}$ toward higher losses. Consequently, the same holds for its convolution $p_{\text{il,w}}^{*^{d-1}}(x)$, the distribution of the sum of losses inflicted by $d - 1$ independent neighbors. Since $F_{\Theta(d)}$ is monotonously increasing as cumulative distribution function we can conclude from Eqs. (A1) and (A2) that $G_d(\mathbf{v}) \leqslant G_d(\mathbf{w})$ as was to be shown.

The essence of the proof is that the increase of the failure probability of a neighbor with degree k also increases the probability that a loss is inflicted to any of its neighbors. This higher probability of a loss can only increase the failure risk (of each node or neighbor).

APPENDIX B: APPROXIMATION OF THE CONVOLUTION

For the DD it is necessary but computationally expensive to calculate the convolution $p_{\text{imp}}^{*^n}$, where

$$p_{\text{imp}}\left(\frac{1}{k}\right) = \mathbb{P}(s_{\text{nb}} = 1|k)\frac{kp(k)}{z\pi}$$

denotes the probability of a loss $L(n = 1) = 1/k$ caused by the failure of one single neighbor with degree k. Given that n neighbors have failed, a node faces a loss $L(n)$, which is a random variable following the law $p_{\text{imp}}^{*^n}$.

However, the number of values l with nonzero probability mass that $L(n)$ attains, which are of relevant size for good accuracy, as well as the number of accumulation points grow exponentially with the order n. But, in the end we are only interested in calculating the failure probability $\sum_l p_{\text{imp}}^{*n}(l)F_\Theta(l)$ of a node. So it suffices to compute $p_{\text{imp}}^{*^n}$ accurately on an interval $[0,b] \subset \mathbb{R}$, where the threshold distribution F_Θ is effectively smaller than 1. For $l \in \mathbb{R}/\,[0,b]$ outside of this interval (which means that $l > b$ since $l \geqslant 0$) we consider the summand

$$\sum_{l>b} p_{\text{imp}}^{*n}(l)F_\Theta(l) \simeq 1 - \sum_{l \leqslant b} p_{\text{imp}}^{*n}(l),$$

since we can approximate $F_\Theta(l)$ by 1 in this region.

We vary the parameters μ and σ of the threshold distribution between 0 and 1 so that we can safely set $b = 5$. Next, we partition $[0,b]$ into J small intervals $I_j =](j - 1)h, jh]$ of length h, with $j = 1,\ldots,J$. For small enough h (here $h = 10^{-5}$) it is numerically precise enough to assume the approximation of $p_{\text{imp}}^{*^n}$ to be constant on each I_j or having its probability mass in an interval I_j concentrated in one point in I_j.

1. Convolution with the help of FFT

In the standard (and faster) algorithm that we use, we simply bin p_{imp} for DD to $[0,b]$ by

$$\hat{p}_{\text{imp}}(jh) := \begin{cases} \sum_{k:(j-1)h < \frac{1}{k} \leqslant jh} p(k) & j = 1,\ldots,J, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we apply the fast Fourier transformation (FFT) [38,39], take the nth power of the resulting distribution and transform it back to obtain $\hat{p}_{\text{imp}}^{*^n}$.

This is numerically accurate enough for the calculation of the final fraction of failed nodes ρ. But sometimes it fails in deducing the correct shape of $\mathbb{P}(s = 1|k)$. For this purpose we have implemented a more precise alternative.

2. Alternative convolution algorithm

Here we assume $\hat{p}_{\text{imp}}^{*^n}$ to have its probability mass in an interval I_j to be uniformly distributed on I_j. For any (discrete) probability distribution p_X we can define an approximation function $a(\cdot)$ that bins a (discrete) probability distribution p_X

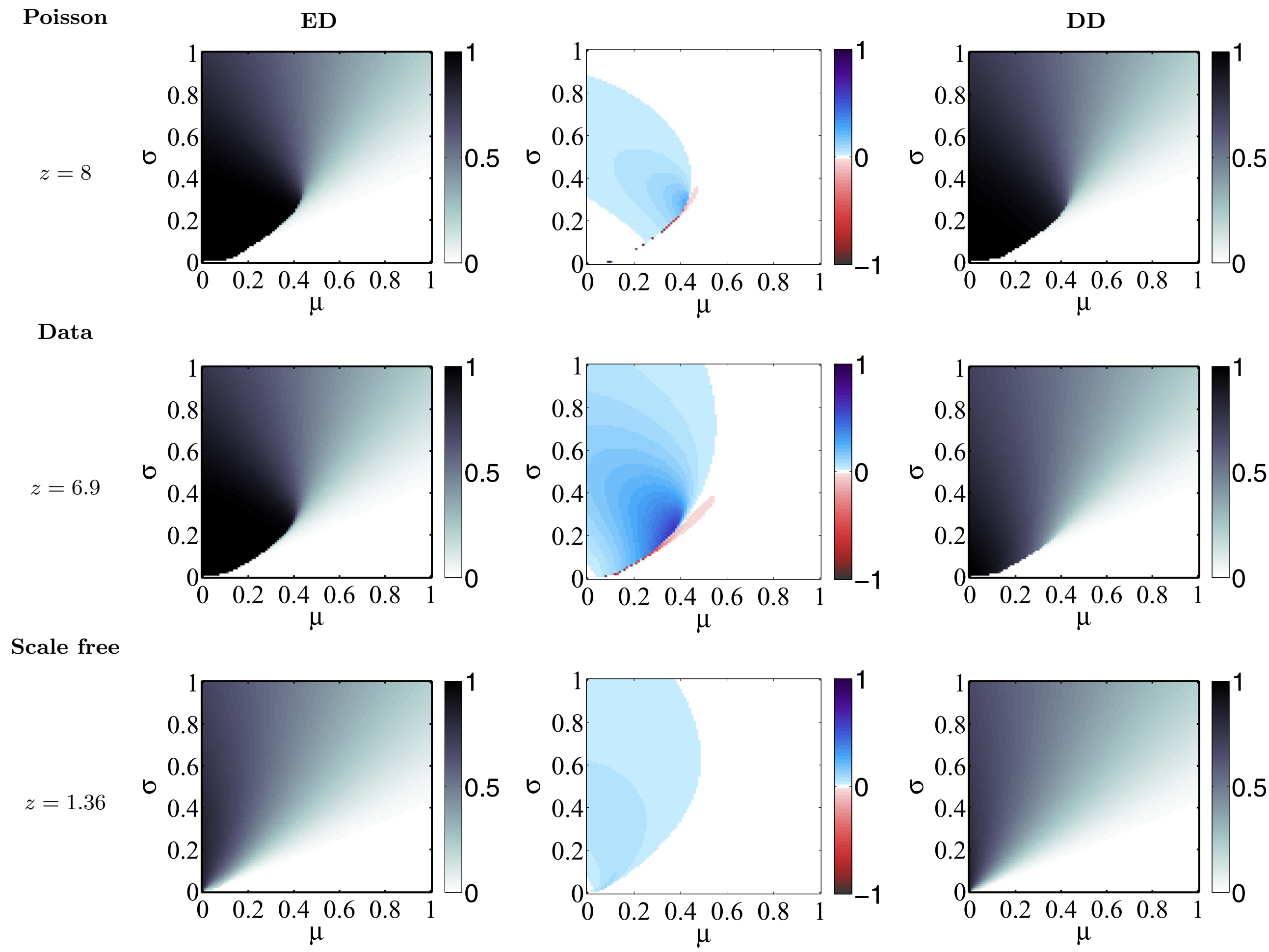

FIG. 7. Phase diagrams for the final fraction of failed nodes ρ calculated numerically for different degree distributions, diversification variants, and their differences. The thresholds Θ are normally distributed with mean μ and standard deviation σ ($\Theta \sim \mathcal{N}(\mu,\sigma^2)$). The fraction of initially failed nodes is given by $F_\Theta(0)$. The darker the color the higher is the systemic risk. First row: Poisson distribution with parameter $\lambda = 8$ and cutoff degree $c = 50$ for the ED (left) and DD (right). The middle shows their difference $\rho^{(\mathrm{ED})} - \rho^{(\mathrm{DD})}$. Second row: Degree distribution of the Italian interbank lending network on October 1, 2002, for the ED (left) and DD (right). The middle shows their difference $\rho^{(\mathrm{ED})} - \rho^{(\mathrm{DD})}$. Third row: Scale-free distribution with exponent $\gamma = 3$ and maximal degree $c = 200$ for the ED (left) and DD (right). The middle shows their difference $\rho^{(\mathrm{ED})} - \rho^{(\mathrm{DD})}$.

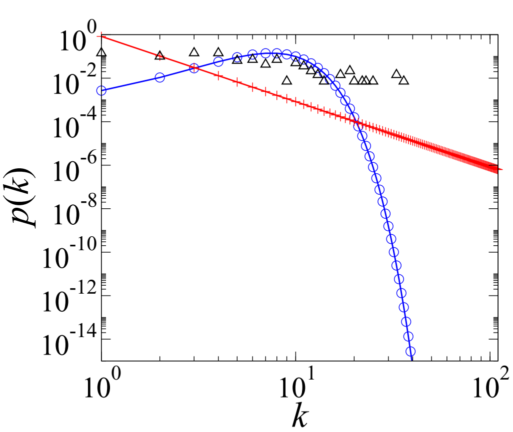

FIG. 8. The studied degree distributions: Poisson distribution with parameter $\lambda = 8$ and cutoff degree $c = 50$ (blue), scale-free distribution with exponent $\gamma = 3$ and maximal degree $c = 200$ (red), and degree distribution measured from a snapshot of the Italian interbank market on October 1, 2002 (black), in log-log scale.

in this way to the intervals $I_1, \ldots, I_J$. We set for $x \in [0,b]$,

$$a(p_X)(x) := \sum_{j=1}^{J} \frac{x-(j-1)h}{h} \mathbb{1}_{\{(j-1)h<x\leqslant jh\}} \times \sum_{y:(j-1)h<y\leqslant jh} p_X(y).$$

Thus, we get for p_{imp}

$$a(p_{\mathrm{imp}})(x) := \sum_{j=1}^{J} \frac{x-(j-1)h}{h} \mathbb{1}_{\{(j-1)h<x\leqslant jh\}} \times \sum_{k:(j-1)h<\frac{1}{k}\leqslant jh} \frac{\mathbb{P}(s_{\mathrm{nb}}=1|k)p(k)k}{z\pi}.$$

This is the initial distribution of an iterative approximation algorithm in which we compute an approximation $\widehat{p_{\mathrm{imp}}^{*^n}}$ of the nth convolution of p_{imp} in the nth step by convoluting first exactly $\widehat{p_{\text{imp}}^{*^{n-1}}}$ of the previous step with the nonapproximated p_{imp}. Afterwards we bin the result of the convolution by $a(\cdot)$ to the intervals I_j again,

$$\widehat{p_{\text{imp}}^{*^{n}}} := a\left(\widehat{p_{\text{imp}}^{*^{n-1}}} * p_{\text{imp}}\right),$$

with

$$\widehat{p_{\text{imp}}^{*^{1}}} := a(p_{\text{imp}}).$$

APPENDIX C: SYSTEMIC RISK FOR FURTHER TOPOLOGIES

We calculate the final fraction of failed nodes for Poisson random graphs and scale free networks with degree distributions

$$p_P(k) := \frac{1}{S_P}\frac{\lambda^k}{k!}, \quad p_S(k) := \frac{1}{S_S}\frac{1}{k^\gamma}$$

for $k \in \{1, \cdots, c\}$ with normalizing constants

$$S_P := \sum_{k=1}^{c} \frac{\lambda^k}{k!} \quad \text{and} \quad S_S := \sum_{k=1}^{c} \frac{1}{k^\gamma},$$

and additionally for a degree distribution measured from a snapshot of the Italian interbank market on October 1, 2002, as published in Ref. [43] and shown in the second row of Fig. 7. The last case we present as illustration of a real world system. Figure 8 depicts the studied degree distributions.

We do not claim a deeper financial interpretation, since this would require incorporating empirical weights as well. This is outside the scope of this paper, which studies the effect of basic diversification strategies.

We observe that the DD variant leads to lower systemic risk for most threshold parameters. Still, close to the sudden regime shift, the ED variant can expose the system to lower risk. This can also be observed for the degree distribution measured from interbank lending data. For other threshold parameters, the DD variant is especially effective for systemic risk reduction in comparison with ED.

The consistently observed regime shift vanishes in the case of both model variants for small connectivity. Scale-free random graphs with $z = 1.36$ show only continuous changes of the final fraction of failed nodes with respect to the threshold parameters.

APPENDIX D: SYSTEMIC RISK FOR NET EXPOSURES

Last, we demonstrate that our analytic approach is not limited to undirected networks.

In many real-world applications, only a net exposure leads to a loss. For example, netting agreements in financial interbank lending networks or flow cancelations leads to effective weights

$$w_{ji}^{\text{eff}} := \max\{w_{ji} - w_{ij}, 0\}.$$

Thus, there exists only an exposure in the direction of the higher weight between two nodes i and j.

In the case of the DD variant, the effective link points from a node with higher degree k_i to a node with lower degree k_j. We have $w_{ji}^{\text{eff}} = 1/k_j - 1/k_i$ and $w_{ij}^{\text{eff}} = 0$, if $k_j < k_i$.

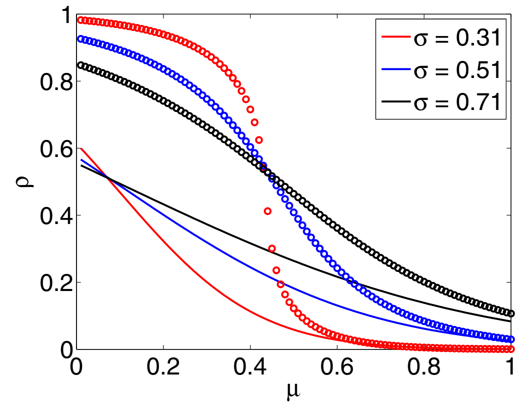

FIG. 9. Final fraction of failed nodes ρ in case of the DD variant for random networks with Poisson degree distribution parametrized by $\lambda = 8$ and cutoff $c = 50$. The thresholds are normally distributed with varying mean μ and standard deviation $\sigma = 0.31$ (red), $\sigma = 0.51$ (blue), and $\sigma = 0.71$ (black). Circles belong to weights defined as w_{ji}^{DD}, while solid lines belong to net exposures w_{ji}^{eff}.

Consequently, Eq. (22) reads now as

$$\mathbb{P}(s = 1|k,\boldsymbol{k_n})^{(\text{DD,eff})} = F_{\Theta(k)}\left(\sum_{j=1}^{n} \max\left\{\frac{1}{k_j} - \frac{1}{k}, 0\right\}\right).$$

To save the computational effort of adding zero weights, we can equivalently restrict the sum only to failed neighbors that have a degree k_j that is smaller than k. A neighbor is failed and exposes a node with degree k to a nonzero loss with probability

$$\pi_k := \sum_{d=1}^{k-1} \frac{p(d)d}{z}\mathbb{P}(s_{\text{nb}} = 1|k).$$

The failure probabilities of a neighbor with given degree solves then the coupled fixed point equations,

$$\mathbb{P}(s_{\text{nb}} = 1|k) = \sum_{n=0}^{k-1} b(n,k-1,\pi_k) \sum_{l} p_{k,\text{eff}}^{*n}(l) F_\Theta(l),$$

where the effective impact distribution $p_{k,\text{eff}}$ considers only nonzero weights,

$$p_{k,\text{eff}}\left(\frac{1}{d} - \frac{1}{k}\right) = \frac{p(d)d}{z\pi_k}\mathbb{P}(s_{\text{nb}} = 1|d),$$

for all $d < k$.

In this setting, the final cascade size ρ is smaller than for the undirected case, since in total the exposures are smaller, as can be seen in Fig. 9.

[1] Edited by A. Garas, *Interconnected Networks* (Springer International Publishing, Switzerland, 2016).

[2] I. Goldin and T. Vogel, Global governance and systemic risk in the 21st Century: Lessons from the financial crisis, Global Policy **1**, 4 (2010).

[3] J. E. Stiglitz, *Globalization and Its Discontents* (W. W. Norton & Company, USA, 2002).

[4] F. Schweitzer, G. Fagiolo, D. Sornette, F. Vega-Redondo, A. Vespignani, and D. R. White, Economic networks: The new challenges, Science **325**, 422 (2009).

[5] A. Garas, P. Argyrakis, and S. Havlin, The structural role of weak and strong links in a financial market network, Eur. Phys. J. B **63**, 265 (2008).

[6] A. G. Haldane and R. M. May, Systemic risk in banking ecosystems, Nature **469**, 351 (2011).

[7] Charles D. Brummitt, Raissa M. D'Souza, and E. A. Leicht, Suppressing cascades of load in interdependent networks, Proc. Natl. Acad. Sci. U.S.A. **109**, E680 (2012).

[8] F. Allen and D. Gale, Financial contagion, J. Polit. Econ. **108**, 1 (2000).

[9] F. Allen, E. Carletti, and R. Marquez, Credit market competition and capital regulation, Rev. Financial Studies **24**, 983 (2011).

[10] S. Battiston, D. D. Gatti, M. Gallegati, B. Greenwald, and J. E. Stiglitz, Liaisons dangereuses: Increasing connectivity, risk sharing, and systemic risk, J. Econ. Dynam. Control **36**, 1121 (2012).

[11] P. Gai and S. Kapadia, Contagion in financial networks, Proc. R. Soc. A **466**, 2401 (2010).

[12] J. P. Gleeson and D. Cahalane, Seed size strongly affects cascades on random networks, Phys. Rev. E **75**, 056103 (2007).

[13] T. Roukny, H. Bersini, H. Pirotte, G. Caldarelli, and S. Battiston, Default cascades in complex networks: Topology and systemic risk, Sci. Rep. **3**, 2759 (2013).

[14] D. J. Watts, A simple model of global cascades on random networks, Proc. Natl. Acad. Sci. U.S.A. **99**, 5766 (2002).

[15] N. Arinaminpathy, S. Kapadia, and R. M. May, Size and complexity in model financial systems, Proc. Natl. Acad. Sci. U.S.A. **109**, 18338 (2012).

[16] R. Pastor-Satorras and A. Vespignani, Immunization of complex networks, Phys. Rev. E **65**, 036104 (2002).

[17] M. Molloy and B. Reed, A critical point for random graphs with a given degree sequence, Random Struct. Alg. **6**, 161 (1995).

[18] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Random graphs with arbitrary degree distributions and their applications, Phys. Rev. E **64**, 026118 (2001).

[19] S. Melnik, A. Hackett, M. A. Porter, P. J. Mucha, and J. P. Gleeson, The unreasonable effectiveness of tree-based theory for networks with clustering, Phys. Rev. E **83**, 036112 (2011).

[20] P. S. Dodds and J. L. Payne, Analysis of a threshold model of social contagion on degree-correlated networks, Phys. Rev. E **79**, 066115 (2009).

[21] J. Payne, P. Dodds, and M. Eppstein, Information cascades on degree-correlated random networks, Phys. Rev. E **80**, 026125 (2009).

[22] H. Amini, R. Cont, and A. Minca, Resilience to contagion in financial networks, Mathematical Finance **26**, 329 (2013).

[23] S. Battiston, D. D. Gatti, M. Gallegati, B. C. N. Greenwald, and J. E. Stiglitz, Credit default cascades: When does risk diversification increase stability? J. Financial Stabil. **8**, 138 (2012).

[24] H. Amini, R. Cont, and A. Minca, Stress testing the resilience of financial networks, Int. J. Theor. Appl. Finance **15**, 1250006 (2012).

[25] M. Catanzaro, M. Boguñá, and R. Pastor-Satorras, Generation of uncorrelated random scale-free networks, Phys. Rev. E **71**, 027103 (2005).

[26] T. R. Hurd and J. P. Gleeson, On Watts' cascade model with random link weights, J. Complex Networks **1**, 25 (2013).

[27] J. Lorenz, S. Battiston, and F. Schweitzer, Systemic risk in a unifying framework for cascading processes on networks, Eur. Phys. J. B **71**, 441 (2009).

[28] H. E. Daniels, The statistical theory of the strength of bundles of threads. I, Proc. R. Soc. London A, **183**, 405 (1945).

[29] D.-H. Kim, B. J. Kim, and H. Jeong, Universality Class of the Fiber Bundle Model on Complex Networks, Phys. Rev. Lett. **94**, 025501 (2005).

[30] C. J. Tessone, A. Garas, B. Guerra, and F. Schweitzer, How big is too big? Critical shocks for systemic failure cascades, J. Stat. Phys. **151**, 765 (2013).

[31] A. E. Motter, Cascade Control and Defense in Complex Networks, Phys. Rev. Lett. **93**, 098701 (2004).

[32] I. Simonsen, L. Buzna, K. Peters, S. Bornholdt, and D. Helbing, Transient Dynamics Increasing Network Vulnerability to Cascading Failures, Phys. Rev. Lett. **100**, 218701 (2008).

[33] A.-L. Barabási, Scale-free networks: A decade and beyond, Science **325**, 412 (2009).

[34] G. Caldarelli, *Scale-Free Networks: Complex Webs in Nature and Technology* (Oxford University Press, Oxford, 2007).

[35] R. Cohen and S. Havlin, *Complex Networks: Structure, Robustness and Function* (Cambridge University Press, Cambridge, 2010).

[36] M. E. J. Newman, *Networks: An Introduction* (Oxford University Press, Oxford, 2010).

[37] Definition of the partial ordering: Two vectors $\mathbf{x}, \mathbf{y} \in [\mathbf{0},\mathbf{1}]^{\mathbf{c}}$ are ordered as $\mathbf{x} \leqslant \mathbf{y}$, if and only if $x_i \leqslant y_i$ holds for all their components $i \in I$.

[38] M. Frigo and S. G. Johnson, The design and implementation of FFTW3, Proc. IEEE **93**, 216 (2005).

[39] P. Ruckdeschel and M. Kohl, General purpose convolution algorithm in S4-classes by means of FFT, J. Stat. Softw. **59**, 1 (2014).

[40] P. Erdos and A. Renyi, On random graphs I, Publ. Math. Debrecen **6**, 290 (1959).

[41] M. Boss, H. Elsinger, M. Summer, and S. Thurner 4, Network topology of the interbank market, Quant. Finance **4**, 677 (2004).

[42] R. Cont, A. Moussa, and E. B. Santos, *Network Structure and Systemic Risk in Banking Systems* (Cambridge University Press, Cambridge, 2013).

[43] G. De Masi, G. Iori, and G. Caldarelli, Fitness model for the italian interbank money market, Phys. Rev. E **74**, 066112 (2006).