# Trade Credit Networks and Systemic Risk

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## 1 Introduction

Credit is extended by banks to firms (loans), by one bank to another (interbank credit) and by one firm to another (trade credit). As a result, there is a *network* of credit relationships among firms, among banks and between firms and the banking system.

Credit relations create value but also financial dependency. Therefore, for a node in the credit network having many links is a way to diversify risk but it is also the ground for the so called financial contagion [1].

In particular, some of these credit relationships are between firms or institutions in different countries and thus connect national credit networks in a world wide network. The possibility of a systemic crisis affecting the whole or a significant part of a credit network raises growing regulatory concern and it is the responsibility of policy makers to ensure that adequate fire walls are in place in order to prevent the spill over of crisis across institutions and firms [18].

An important and open debate, with major policy implications, concerns whether or when higher network density (in other words more links in the network) leads to lower or higher systemic risk (in the sense of probability of joint failures causally related).

The dominant neoclassic approach in economics typically assumes (1) equilibrium and/or (2) indirect interaction through price. The failure of coordination which is likely to arise in a decentralized market economy is simply assumed away. [6].

This approach to economic theory has been recently challenged by the approach based on heterogeneous interacting agents, which conceives the economy as a complex system. The starting point of this approach is that while prices surely play a fundamental role, the price mechanism can work well only if information is perfect and markets are complete. If this is not the case, i.e., if the future is uncertain, it is not possible to ignore direct interactions and co-ordination mechanisms that arise in spatio-temporal way – i.e. supply chains, communication, imitation, learning, trust and credit relationships. In this context, complex patterns of heterogeneous agents' interactions at the micro level lead to the emergence of statistical regularity at the macroeconomic level through a self-organised process. A pioneering work applying a complex systems approach to the macroeconomic impact of a production network dates back to the early 90' [2]. There, it was shown that local uncorrelated fluctuations can nevertheless generate, through interaction, large aggregate output fluctuations. Concerning credit networks, the view resulting from the dominant approach tends to see more dense networks as more stable. In this chapter, we show how the interacting agents approach results in a different view.

While banks-firms credit relationships have been extensively studied since long ago in the economic literature (for an overview, see [5]), a recent interesting line of research has analysed phenomena of financial contagion in interbank credit [1, 18]. Finally, trade credit , is less investigated but yet an important part of the network of credit relationships. It represented, for instance, one half of the short term liabilities of the corporate sector in 2004 in the U.S. [3]. Moreover, trade credit is largely used as collateral in bank borrowing, especially by small and medium sized firms. In the U.S., lines of credit secured by accounts receivables represented approximately one quarter of total bank loans in 1998 [8]. In Italy, loans secured by receivables were 22% of total loans and 54% of short term loans in 2002 [12]. In the theoretical literature, [7] emphasize the role of trade credit as a propagation mechanism (the so called *balance-sheet contagion*) , while the dynamics of credit chains has been investigated by [3].

From the point of view of complex systems, few important works have applied the concept of self-organized criticality (see also below) to the context of interbank markets [15, 19].

However, the issue of systemic risk in credit networks remains to some extent underresearched, both at the theoretical and empirical levels.

In this chapter, we present a model recently introduced in [14, 16] and we discuss the features of a networked economy in which $N$ firms are organised in $M$ production levels. Each firm at a certain level is supplied by a subset of firms in the upper level (suppliers) and supplies a subset of the firms in the lower level (customers). The bottom level consists of retailers, i.e., firms that sell in the consumer market. The top level consists of firms that provide primary goods to the other firms. Firms are connected by means of two mechanisms: (i) the output of supplier firms is an input for customer firms; (ii) supplier firms extend trade credit to customers (as it is typically the case in reality).

However, in the model, the trade credit contract is only implicitly sketched: we neither design the optimal trade credit scheme nor look for the optimal amount of trade credit a customer firm should require. Instead, we focus on the mechanisms of propagation of bankruptcy . When a firm is unable to reimburse debt, it goes bankrupt. This may happen as a result of one of (or any combination of) three mechanisms: (1) there is a production default in the firm (production is lost and so is profit, while the firm still has to pay for input and processing); (2) some customers are not able to pay; (3) some suppliers are not able to deliver the agreed input (in this case the firm does not bear the cost of input but still does bears some cost due the fact the resources were allocated in view of processing that input).

Thus, the failure to fulfill debt commitments by a customer may hamper the solvency of the supplier, who may become unable in turn to pay its own suppliers located in the upper level, which may lead to a chain of similar failures (domino effect) and in extreme cases result in bankruptcy avalanches. When a firm goes bankrupt, in fact, the probability of bankruptcy in connected firms increases, yielding clustered fluctuations in the number of failing firms. In other words, a single bankruptcy may have systemic repercussions through an avalanche of bankruptcies.

In this context, having many customers and many suppliers is a way for the firm to diversify the risk of defaulting payment or delivery. If the network is dense enough, the default of a firm in paying its debt doesn't cause any other default. For instance, if every firm has $k$ customers (with similar volume of orders), the default of one customer in paying causes a unexpected relative decrease in profit of order $\frac{1}{k}$. The larger $k$ the smaller the unexpected loss.

However, in presence of externalities , the loss caused by a defaulted payment or delivery may be amplified through the network. Some multi-agent models of financial fragility have been able to account for this effect. In [4] a single bankruptcy may have systemic repercussions: in fact, the banking system reacts to the bankruptcy by restraining the supply of credit and pushing up the interest rate to all firms. The increase in the interest rate may cause some other bankruptcies and thus trigger an avalanche of bankruptcies. Such models incorporate only the indirect interaction among firms that takes place through the endogenous determination of the interest rate on bank loans.

In the present model, instead, direct interaction among firms takes place through supply and extension of trade credit which is also subject to an interest rate. If the interest rate is dynamic and depends on the change of growth rate of the firm itself and its neighbours, then losses can be amplified through credit relations. Under such conditions, increasing the network density, while decreasing the shocks to individual firms, it may also increase the systemic risk, thus inducing a trade-off between individual risk diversification and global instability.

This result is consistent with a recent work on failure avalanches in complex networks [17] which has pointed out the role of the interplay of two opposing mechanisms: diffusion and contagion. On one side, when energy diffuses from a node to its $k$ neighbours, the energy received by each neighbour is of order $\frac{1}{k}$ of the initial one. On the other side, in a contagion process, nodes have discrete states and with a certain probability switch from one to the other when a neighbour has changed state. Therefore, if both mechanisms are at work at the same time, then increasing the density of the network, the impact of a change of state of a given node on the neighbours first decreases due the diffusion and then increases due to the contagion. The work by [17] bridges two strains of work in the physics literature on complex systems: the models on cascading phenomena on one side and those on epidemic spreading on the other side.

Avalanches of events in networks have been studied extensively in the context of self-organised criticality (SOC) and in particular in models inspired to the sand pile model [9] and the fiber bundle model [13]. In all these models, an event on a node of the network ( a "toppling") transfers energy to neighbouring nodes, possibly triggering their toppling. Each node is associated with one state variable, which depends on the toppling of the neighbours and causes the node to topple when it reaches a given threshold. In the SOC models there is a slow build-up mechanism (flow of sand, increase of the force on the bundle) acting everywhere in the system and decreasing over time the distance of the state variable of the nodes from the toppling threshold. Without this build-up mechanism the system would not become critical and the network density would increase the resilience of the system. On the other hand, the works on epidemic spreading have shown that if the network is dense or there are hubs the onset of the epidemic phase is facilitated [11].

Overall, the investigation on failure propagation in the context of credit networks of firms deserves more attention. Besides the work of [14, 16] presented here, [10] have recently studied a similar model where, however, there is no credit and cost is only proportional to delivered input, so that bankruptcy occurs only as result of production defaults. In fact, such model addresses a different issue related to the emergence of activity patterns in geographical economics.

The rest of the chapter is organised as follows. In section 2 we describe a modelling framework for networks of firms engaged in supplier-customer relations. We discuss the properties of a specific model in section 2.10, reporting some analytical results and some computer simulations. Some conclusions are drawn in section 3.

# 2 The Model

## 2.1 Economic Environment

The economy consists of $N$ firms organised in $M$ production levels. We will denote firms with indices $i, j, k, l, ...$ and levels with indices $J, K, L, ....$ We adopt the convention that production takes place along the vertical axis in downwards direction. The structure of the connections defines the production network as in the example shown in figure 1, in which arrows represent supply of goods (supply proceeds downwards, while money moves upwards).

In the example, each node has the same number of links $k$, but in general this could be from any distribution and, besides, the number of incoming links

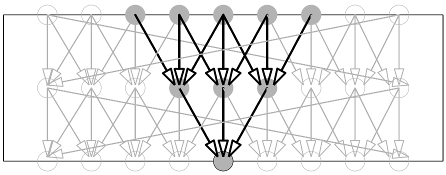

**Fig. 1.** Example of structure for the production network. The direction of production is from top to bottom. Each firm in a level receives goods from a subset (3 in this case) of firms from the upper level. The top level consists of primary producers. Layer 3 (from the top) consists of firms that sell in the consumer market (retailers). We have highlighted in dark gray the set of all suppliers upward from a given retailer (*in green*)

do not have to even outgoing links. Each firm in a level $K$ is supplied by a subset of firms in the upper level $K-1$ and in turn supplies a subset of the firms in the lower level $K+1$. The bottom level $K=M$ represents firms that sell in the consumer market (retailers). The top level $K=1$ represents primary producers. Firms are connected to each other through two mechanisms:

1. A firm asks for inputs from the suppliers in order to produce output.
2. A firm asks for payments from the customers in order to realize profit.

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The output of each level $K$ is produced by processing the input from the previous level $K-1$. Output is qualitatively different from input. For the sake of simplicity, we assume the following linear technology:

$$Y_i^{(K)} = \sum_{j \in V_i^S} Q_{ij}^{(K,K-1)} Y_j^{(K-1)} \tag{1}$$

where $Y_i$ is the output of firm $i$, $S_i$ is the set of suppliers of firm $i$, and $Q_{ij}$ represents the fraction of the total output of firm $j$ that firm $i$ uses to produce its own output. In other words, $Q$ is the input-output matrix and for any $K$, it follows that:

$$\sum_{i\ \in\ \text{level}\ K} Q_{ij}^{(K,K-1)} = 1, \forall j \in \text{level } K-1 \tag{2}$$

## 2.2 Timing

We have to model the fact that over time, firms decide their desired amount of production, send orders, produce, deliver to customers and pay suppliers. We assume that time is discrete and divided into periods, each period including the following events for all firms: At the beginning of each period (or time step) $t$, orders flow upwards; then production and delivery flow downward. At the end of the period, money flows upward.

In greater detail, at the beginning, all firms in the bottom level $M$ determine their *desired output*, based on the demand they face on the market and their *production capacity*, and then send orders to the upper level $M-1$. Afterwards, all firms in level $M-1$ determine their desired output based on the demand they face from their customer firms in level $M$. One after another, all levels do the same, up to level 1 (primary producers). Once the desired output is known, firms can compute their *expected output*, based on the expected output of the suppliers, which they communicate to the firms downward. This allows customer firms to allocate the necessary resources and premises to process the inputs they will receive.

At this point, production starts in level 1 and proceeds downward one level after the other, as each firm needs the input from its suppliers in order to produce. Output produced by a firm is delivered to customers on the basis of full trade credit; we rule out the possibility of inventory accumulation. When production reaches the bottom level, products are fully sold in the consumer market.

At the end of the period, a sequence of payments proceeds upwards from the retailers up to the primary producers. At each level, each firm pays its suppliers upstream only after having been paid by its customers. If costs exceeds revenues, the firm goes bankrupt and does not pay the suppliers in the current period. Moreover, the firm stops production for a number $\tau$ of periods in the future, after which it is replaced by a new firm endowed with an assigned initial value of production capacity. During those $\tau$ periods, the suppliers of that firm do not receive orders from it, nor do the customers receive production from it. Therefore, bankruptcy at the end of period $t$ results not only in disruption of payments but also in a temporary local disruption in the production chain which is repaired in period $t+\tau+1$.

## 2.3 Remarks

The structure of the connections does not change during the process. This means that when a firm goes bankrupt, its customers do not create new links with other suppliers. This follows from the assumption of prohibitively high costs of establishing relations with new suppliers. So far, we have described a general framework, while the mechanisms involved can be specified in several ways (for example, we have to specify the dynamics of price, profit and net worth). However, some of the results presented in this paper do not depend on the specification of such mechanisms. Therefore, the present structure is a candidate for a class of models sharing similar behavior, in particular, concerning the conditions for the occurrence of avalanches of bankruptcies which are analysed in section 2.10. In the following, we provide a detailed description of a simple version of the model and a discussion of its limitations. In any period $t$ each firm $i$ is endowed with a level of real net worth $A_i(t)$, defined as the stock of the firm's assets in real terms, that has been financed only through net profits (we assume complete equity rationing).

### 2.4 Desired Output

Firm $i$ at level $K$ determines at time $t$ its *desired output*, $Y_i^{(d,K)}$. This depends on the orders received from level $K+1$, with the constraint of its production capacity that we assume to be proportional to net worth $A_i^{(K)}$ by a constant $\theta > 0$ (as stated in eq. 3). Therefore, capacity is financially constrained as, for instance, in Greenwald and Stiglitz,(1993) and in related work by Delli Gatti et al., (2005). As in Greenwald and Stiglitz we can conceive of $\theta A_i^{(K)}(t)$ as the optimal (i.e., maximizing expected profit) output in the presence of bankruptcy costs.

Hence, desired output is defined as follows:

$$Y_i^{(d,K)}(t) = \min\{\theta A_i^{(K)}(t), \sum_{\mathrm{j} \in V_i^C} O_{ij}^{(K,K+1)}(t) Y_j^{(d,K+1)}(t)\} \tag{3}$$

In the equation above, $V_i^C$ is the set of customers of firm $i$, $O^{(K,K+1)}$ is the order matrix describing the orders from level $K+1$ to $K$, and in particular $O_{ij}^{(K,K+1)}$ is the fraction of the total supply needed by firm $j$, that firm $j$ orders to firm $i$. In matrix notation we can write:

$$Y^{(d,K)}(t) = \min\{\theta A^{(K)}(t), O^{(K,K+1)} Y^{(d,K+1)}(t)\} \tag{4}$$

For level $M$, we assume that at each time step the consumer market absorbs the whole production and therefore:

$$Y^{(d,M)}(t) = \theta A^{(K)}(t) \tag{5}$$

### 2.5 Expected and Effective Output

Once the desired output is known at all levels, firms compute their *expected output*, based on the expected output of the suppliers. Here, "expected" has nothing to do with "expectation value" in statistical sense. A firm $i$ may not be able to fulfill the orders of its customers, either because they exceed its production capacity or because the input from its suppliers is insufficient. As a result, supply can be smaller than the ordered quantity and therefore the *expected output* of firm $i$, $Y_i^{(e)}$, can be smaller than the desired one $Y_i^{(d)}$. In this version of the model, firms have a fixed set of suppliers (the network structure is static) and they cannot look for new suppliers. However, there is some freedom in the way firms decide to place orders to their suppliers, in other words, the way $O_{ij}^{(K,K+1)}$ are determined. This is discussed later on, and plays an important role. The production function of firms is assumed to be linear so that the output of a firm in level $K$ is a linear combination of the input received from the suppliers in level $K-1$. This yields:

$$Y_i^{(e,1)}(t) = Y_i^{(d,1)}(t)$$
$$Y_i^{(e,K)}(t) = \sum_{\mathrm{j} \in V_i^S} Q_{ij}^{(K,K-1)}(t) Y_j^{(e,K-1)}(t) \tag{6}$$

For firms at level 1, the expected output coincides with the desired one, as they do not have suppliers. $V_i^S$ is the set of suppliers of firm $i$, $Q^{(K,K-1)}$ is the input-output matrix describing the transformation of input from level $K-1$ into the output of level $K$. Each entry $Q_{ij}^{(K,K-1)}$ represents the fraction of the total output of firm $j$ that firm $i$ uses to produce its own output. Firms in level 1 are primary producers and do not need any supply, therefore $Y_i^{(e,1)} = Y_i^{(d,1)}$. In matrix notation, the output of any level can be expressed as a function of the output of the first level as follows:

$$Y^{(e,K)}(t)=Q^{(K,K-1)}(t)Y^{(e,K-1)}(t)=Q^{(K,K-1)}(t)\cdot ...\cdot Q^{(2,1)}(t)Y^{(e,1)}(t) \quad (7)$$

The expected output is communicated downward to customers. Any two firms engaged in a supplier-customer relation agree on this amount to be delivered and paid at the end of the period. Customer firms allocate the necessary resources and premises to process the expected input they will receive from suppliers.

At this point, we include in the model some occasional production failures (due, for instance, to technical problems). At each period $t$, with probability $q$, the production of firm $i$ is lost during the processing and no output is delivered to customers. This event occurs independently of the financial state of firms $i$ and this failure lasts only one period. Therefore, we have to rewrite the *effective output* of $i$, $Y_i^{(K)}(t)$, as:

$$Y_i^{(K)}(t) = Y_i^{(e,K)}(t)S_i(t) \quad (8)$$

where $S_j(t) = 1$ with probability $q$ and $S_j(t) = 0$ with probability $1-q$.

## 2.6 Production Costs

The output produced by firm $i$ is sold to the customer at the price $P_i(t)$ (no inventory accumulation). We can think of the price of a firm's output in level $K$ as $P_i(t) = P^{(K)}(t)u_i(t)$ where $P^{(K)}(t)$ is the general price at level $K$ and $u_i(t)$ is the relative price for the output of the single firm. We assume that $u_i(t)$ is a random variable, uniformly distributed in $[1-\delta_P,\, 1+\delta_P]$ and independent of $P^{(K)}(t)$. Therefore, firm $i$ incurs the following cost to get its supply of inputs from level $K-1$:

$$\tilde{C}_i^{(s,K)}(t) = \sum_{\mathrm{j}\,\in\, V_i^S} Q_{ij}^{(K,K-1)} P^{(K-1)}(t) u_j(t) Y_j^{(K-1)} \quad (9)$$

The cost of inputs in real terms is obtained by dividing nominal costs by the level of prices in the level $K$:

$$\begin{aligned} C_i^{(s,K)}(t) &= \frac{P^{(K-1)}(t)}{P^{(K)}(t)} \sum_{\mathrm{j}\,\in\, V_i^S} Q_{ij}^{(K,K-1)}(t) u_j(t) Y_j^{(K-1)}(t) \\ &= c_s \sum_{\mathrm{j}\,\in\, V_i^S} Q_{ij}^{(K,K-1)}(t) u_j(t) Y_j^{(K-1)}(t) \quad (10) \end{aligned}$$

where $c_s$ is defined as the ratio of the price levels at level $K-1$ and $K$ and we assume it to be the same for all $K$.

Firm $i$ also incurs a cost associated with the resources used in processing the input (labour and premises). As for the supply cost, this cost is assumed to be proportional to the expected output through a constant $c_r > 0$. We assume that the resources allocated by the customers of $i$ to process its expected output cannot be dis-allocated within the current time period. Therefore, in case of a production failure of $i$, its customers run a cost proportional to the expected output and not to the effective output:

$$C_i^{(r,K)}(t) = c_r Y_i^{(e,K)} = c_r \sum_{\mathrm{j} \in V_i^S} Q_{ij}^{(K,K-1)}(t) Y_j^{(e,K-1)}(t) \tag{11}$$

Of course, in the case of a production failure by $i$, the customers of $i$ do not incur any supply cost. On the other hand, firm $i$ not only does not receive any payment but has also to pay for the input from its suppliers. The production of firm $i$ resumes at the next time step, if it has survived the shock. In conclusion, the production cost of firm $i$ is the sum of the two terms defined above:

$$C_i^{(K)}(t) = C_i^{(s,K)}(t) + C_i^{(r,K)}(t) \tag{12}$$

## 2.7 Profit and Bankruptcy

In each period, when output is sold in the consumer market and payments start, some firms may realize sales revenue smaller than their supply costs. If this loss is high enough, firms go bankrupt and do not pay their suppliers . Therefore, we have to distinguish between the output delivered by firm $i$ to its customers, $Y_i(t)$, and the output $Y_i^s(t)$ that is actually paid for ("s" for "sold"), at price $u_i(t)$, to firm $i$ by its customers. Profit in real terms is equal to the difference between revenues and costs in real terms.

$$\pi_i^{(K)}(t) = u_i(t) Y_i^{(s,K)}(t) - C_i^{(K)}(t) \tag{13}$$

Profit, which can be negative or positive, incrementally changes the real net worth of the firm:

$$A_i^{(K)}(t+1) = \rho A_i^{(K)}(t) + \pi_i^{(K)}(t) \tag{14}$$

where $1-\rho$ measures a depreciation rate.

We assume that firms go bankrupt when the ratio of profit and net worth becomes smaller than a negative threshold value:

$$\pi_i^{(K)} < -\beta A_i^{(K)} \tag{15}$$

with $1 > \beta > 0$. If a firm goes bankrupt at time $t$, it stops supplying customers and paying suppliers for a number $\tau$ of time steps (referred to as "inactivity time" in the following). During these time steps, neighboring firms are not allowed to look for alternative customers or suppliers, as the network structure is static. Firms can however (at least in some of the scenarios considered in the following) adjust their orders as a function of the production capacity of the suppliers. As this is proportional to net worth, it means that customers order less and less when a supplier's net worth decreases. Once the inactivity time elapsed, the bankrupt firm is replaced by a new firm with the same links as its predecessor and its net worth is re-initialized: $A(t+\tau+1) = A_{entry}$.

### 2.8 Strategies for Placing Orders and Delivery

Although the network is static in this version of the model, and therefore the set of suppliers of a firm is fixed, still there are many possible ways to allocate orders to the suppliers. Consistently with our bounded rationality framework, we consider simple strategies for placing orders and one strategy for delivering. Firm $i$ places orders evenly:

$$O_{ij}^{(K,K+1)}(t) = \frac{1}{|V_i^S|} \tag{16}$$

where $|V_i^S|$ is the cardinality of the set of suppliers $j$ of firm $i$ (notation is consistent with equation 3).

Firm $i$ delivers to each customer $j$ in proportion to its order:

$$Q_{ij}^{(K,K-1)}(t) = \frac{O_{ij}^{(K,K+1)}(t) Y_j^{(K-1)}(t)}{\sum_{l \in V_j^C} O_{lj}^{(K,K+1)}(t) Y_j^{(K-1)}(t)} \tag{17}$$

The equation above satisfies the condition of equation 2. For other possible strategies see ([14])

### 2.9 Generic Properties of the Model

A number of specific models can be investigated within the framework presented so far. In particular, in the presence of delayed payments (trade credit) and costs due to failures in supply (as assumed above), in this model it is possible to have avalanches of bankruptcies originating locally and spreading both upstream and downstream.

If bankruptcies can propagate simultaneously in both directions, then, and only then, are they "reflected" diagonally at each level and the result is a net horizontal propagation, that is perpendicular to the direction of production (figure 2, c-d). The horizontal propagation is important because it is a necessary condition for the spreading of an avalanche to a significant part of the network if the number of layers is much smaller than the number of site positions. This is typically the case in several sectors and was one of the weak points among the condition for the emergence of Self Organized Criticality

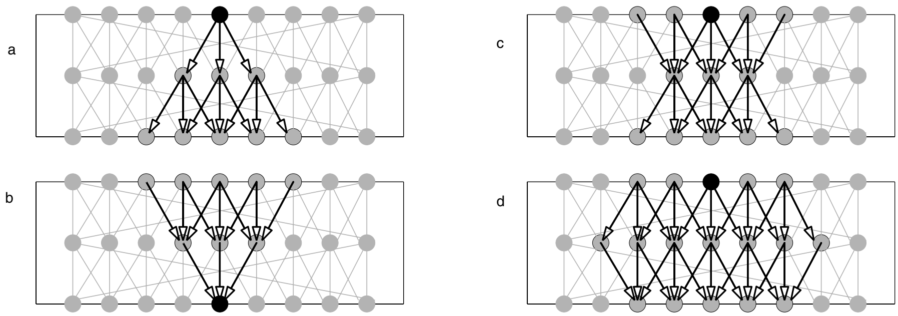


**Fig. 2.** Different modalities of failure propagation. Edges through which failure propagate are in darker gray. The firm triggering the avalanche is represented by the node in dark gray. **a.-b.** Downward and upward propagation of failures. **c.-d.** Horizontal propagation occurs when each level transmits downward but also reflects upwards. In panel c failures have propagated up to two degrees of separation from the initial firm; in panel d up to three degrees

in the work of [2]. In particular, the horizontal axis could also represents a geographical or technological space.

In the following we will speak of *horizontal bankruptcy propagation* to mean the situation in which bankruptcies can propagate potentially to the whole network and not only to the downward/upward cone of firms.

On the contrary, previous models ignore local interaction so that the propagation of bankruptcies is activated only by means of global coupling: the more firms fail, the higher the interest rate for all, hence the more they fail.

The model, as presented so far, reproduces qualitatively important properties of a production network:

1. Spatio-temporal correlation of output, growth and bankruptcies
2. Exponential growth
3. Oscillations of de-trended aggregate output
4. Heterogeneous firm size distribution
5. Exponential probability distribution of aggregate growth (right side)

For a detailed discussion of this properties, see [14]. Varying allocation strategies, dynamics on prices and other parameters one can investigate the role of the main factors involved in models of financial fragility and address the following issues:

1) The role of trade-credit relationships in the propagation of bankruptcies

2) The role of interest rate and policies to prevent the occurrence of large avalanches

3) The role of the structure of the network of interactions

4) Policies to make such structure more robust against large avalanches

In the rest of this chapter, we will focus on the role of the network density in combination with the dynamics of the interest rate, an issue discussed more in depth in [16].

### 2.10 Analysis of a Specific Setting

In order to isolate the impact of network density on the dynamics of avalanches, we will now consider a specific case of the model in which there is no accumulation of net worth ( we set $\rho = 0$ in eq. 14). As a result, aggregate output is no longer growing exponentially and, moreover, firm size distribution does not evolve in time into a skewed distribution. Consistently, we also set $\beta = 0$ in eq. 15 : as firms do not accumulate net worth, they go bankrupt when profit is not positive. Therefore, in this setting, a production default implies also bankruptcy.

Concerning prices, we assume, as discussed in [14], that prices are stochastic and independent, distributed according to a uniform distribution in $[1 - \delta_P, 1 + \delta_P]$. Eq. 15 implies that a firm goes bankrupt if the price falls below a critical value, which can be approximated as:

$$u_i^*(t) = \frac{(c_r - \beta/\theta)Y_i^e(t) + c_s Y_i(t)}{Y_i^s(t)} \tag{18}$$

Because it is $Y_i^s \leq Y_i \leq Y_i^e$, the probability of bankruptcy increases the smaller are, with respect to the expected output $Y_i^e$, $Y_i$ and $Y_i^s$ (see section 2.7).

#### Density Decreases Systemic Risk

For the sake of simplicity, we assume that firms have the same number $k$ of suppliers and customers, and we study the impact of different values of $k$. We consider $M$ production layers, each including the same number $n$ of nodes. If $k = 1$ the network is actually composed of isolated chains, while if $k = n$ all possible links are realized. Clearly, increasing $k$ reduces the fluctuations of input from suppliers due to production default and therefore, the probability of bankruptcy. Consider for simplicity, a two-layer network; for $k = 1$, with probability $q$ a customer is not delivered at all (because the supplier experiences a production default with probability $q$) and goes bankrupt. For large $n$, input delivered to each customer approaches the fraction $1-q$ of the input requested, and thus, if $\frac{c_r}{1-q} + c_s < 1$, the probability of causing bankruptcy of customers to go bankrupt as a result of a production default among suppliers is zero. [4]

[4] Of course, one could assume that firms incorporate probability if defaults in their decision, by ordering $\frac{1}{1-q}$ so to be delivered the exact requested quantity. Instead, as before, for the purpose of this work we assume agents are boundedly rational and do not take into account production defaults in their decisions.

We can make precise predictions on the bankruptcies caused by a production default by computing the profit of the neighbouring firms. In a two layer network, after a production default in a supplier, the profit of each customer is:

$$\pi_j(t) = pY_i^s(t) - c_sY_i(t) - c_rY_i^e(t) = (p - c_s - \frac{k}{k-1}c_r)Y_i(t) \qquad (19)$$

Similarly, after a production default in one customer, the profit of each supplier is:

$$\pi_j(t) = pY_i^s(t) - c_sY_i(t) - c_rY_i^e(t) = (\frac{k-1}{k}p - c_s - c_r)Y_i(t) \qquad (20)$$

In both cases, the increase of $k$ increases the profit and makes a bankruptcy less probable. We can also estimate the profit of firms in case of multiple defaults in the neighbourhood, and thus compute the expected profit in the general case [16]. Overall increasing network density reduces the probability of bankruptcies of individual firms, as well as the probability of joint bankruptcies, in other words, it reduces the systemic risk.

### Systemic Risk in Presence of Positive Feedback

However, if there is a positive feedback of the probability of bankruptcy of a firm $i$ on the cost $i$ faces (namely, that the more a firm is likely to fail the higher the cost it faces), then the effect of network density can be to increase the instability of the system. In fact, in a very dense network, an increase in the average probability of failure would increase the cost of all firms, thus increasing in turn their probability of failure. In a sparse network, the coupling is only local so that the probability of failure may increase somewhere while decreasing somewhere else. In order to investigate quantitatively this issue, we now, consider the cost of the firm to be dependent on the financial state of the firm. The rationale for this is that firms pay an interest rate on the supply they receive (trade credit) and/or on the funds used to pay wages or processing (loan). The interest rate a firm is charged by other firms or by the bank increases (at least within a range) with the financial fragility of the firm itself as its partners need to compensate their risk in extending credit to it (however, when the interest rate is very high, creditor usually don't have an incentive to increase it further, see [5] (chapter 5). In order to capture this effect, we assume that an increase in bankruptcy risk (a decrease in profit) leads to an increase in interest rate and therefore in production costs. As a consequence, the production cost for firm $i$ is multiplied by a factor $\eta$ evolving in time as follows:

$$\eta_i(t+1) = \eta_i(t) + \alpha \cdot \text{sign}(\sum_{j \in V_i} P_j(t) - P_j(t-1))$$
$$\eta_i \in [\eta^{\min}, \eta^{\max}] \qquad (21)$$

where $V_i$ is the neighbourhood of $i$ including $i$ itself, while $\alpha$ is a parameter. The range of variation of $\eta_i$ is bound, corresponding to a minimum and maximum interest rate. The equation above implies that whenever profit decreases/increases among neighbours, cost increases/decreases by a fixed quantity $\alpha$. Other functional dependencies are possible and reasonable, but the important feature here is that the net average change of profit in the neighbourhood causes a discrete change in the cost.

### Understanding the Dynamics in a Simplified Model

A similar dynamics has been recently introduced in [17] in the context of cascades in complex networks . There, agents are associated with a state variable, representing their fragility, that evolves as a function of the neighbours. At each time step, the fragility of each agent receive an i.i.d. shock ( through a normalized stochastic variable $\xi(t)$, with standard deviation equal to 1), which is shared with the neighbours. If, at time $t$ the fragility of agent $i$ exceeds a given threshold $\theta$, the agent fails and the quantity $a$, representing the damage associated with its failure, is distributed to the neighbours (by incrementing their fragility), which may in turn fail. All the toppling events following the initial failure occur at time scale faster than the one for fragility, in other words they all occur before the next time step $t+1$. In formulas the dynamics reads:

$$\phi_i(t+1) = \sum_{j\,\in\,V_i} W_{ij}(\phi_j(t) + \sigma\xi_j(t)) + \alpha \cdot \mathrm{sign}(\sum_{j\,\in\,V_i} W_{ij}(\phi_j(t) - \phi_j(t-1))) \tag{22}$$

where $W$ is a matrix representing interaction among agents, with $\sum_j W_{ij} = 1 \forall i$, and $\sigma$ is a parameter. Additionally, if $\phi_r(t+1) \geq \theta\ \exists r$, then:

1. For all neighbours of each node $r$:
   $\phi_s \rightarrow \phi_s + aW_{sr}$
2. For all such $r$, $\phi_r \rightarrow 0$

3. Repeat until $\phi_i < \theta$ for all $i$ in the system.

In order to understand the onset of instability we analyse the dynamics above in a mean field approximation, in the case $W_{ij} = \frac{1}{k}$. We consider the average fragility in case of large $k$:

$$\Phi(t+1) = \frac{1}{N}\sum_i \phi_i(t+1) \simeq \Phi(t) + \frac{\sigma\xi(t)}{\sqrt{k}} + \alpha \cdot \mathrm{sign}(\Phi(t) - \Phi(t-1)) \tag{23}$$

In the regime where $\frac{\sigma\xi(t)}{\sqrt{k}} \gg \alpha$, the process is dominated by the first term of eq. 24 and it is approximated by a random walk with step of amplitude

$\frac{\sigma}{\sqrt{k}}$, thus decreasing with $k$. In particular, if the first term dominates, the difference $\Phi(t) - \Phi(t-1)$ is positive or negative with equal probability and thus the second term does not contribute any systematic drift. In this regime, increasing $k$ makes the step of the random walk smaller and thus decrease the probability to hit the threshold.

If $\frac{\sigma}{\sqrt{k}} \ll \alpha$, then expressing $\Phi(t)$ in terms of $\Phi(t-1)$, we have:

$$\begin{aligned} \text{sign}(\Phi(t) - \Phi(t-1)) = \text{sign}(\frac{\sigma\xi(t)}{\sqrt{k}} & +\alpha(\text{sign}(\Phi(t-1) - \Phi(t-2)))) = \\ & +\alpha(\text{sign}(\Phi(t-1) - \Phi(t-2)))) \end{aligned} \tag{24}$$

The expression above is always true if the distribution of $\xi(t)$ has a limited support and $\alpha$ is larger than the right limit of such support, otherwise it is true with a certain probability that can be computed. Therefore, in this regime the average fragility tends to keep moving in the direction it is already moving. Because $\Phi$ is repelled from 0 by construction, in the limit of large $k$, it moves upwards with constant slope and periodically it hits the threshold and is then reset to 0. When the average fragility hits the threshold and the individual fragility trajectories are sufficiently close, then one or few failures cause an avalanche involving the whole system.

The argument suggests therefore that, increasing the density of network in the system described by eq. 22, the probability of failures first decreases and then increases. In other words there is a trade-off between diversifying the risk by sharing the shocks with many other agents and the systemic risk resulting from the synchronization of the fragility trajectories. For a more formal analysis see [17].

The argument above suggests that a similar result should also hold for the economic model presented in this chapter. The reader may notice that we have inferred an important property of a fairly complicated economic model from a basic argument, based on a mean field approximation of a simplified model that captures some essential dynamical features of the original model. In the next section we will examine the results of computer simulations of the original model.

### 2.11 Results of Computer Simulations

In this section, we compare the time evolution of some quantities measured on a network of 2 production layers, with three different values of connectivity degree, $k = 1, 5, 20$. Unless specified otherwise, results reported in this work are obtained with constant price (interval width $\delta_P = 0$), production default probability $q = 0.03$, $c_s = c_r = 0.3$, inactivity time $\tau = 1, 2, 3$ with equal probability. Firms are endowed with constant value of net worth $A_i(t) = A_{init} = 1\,\forall i$. As explained above the bankruptcy threshold is set as $\beta = 0$ and the depreciation factor is set as $\rho = 0$ (which yields a depreciation rate $1 - \rho = 1\%$ ). Finally, the value of $\alpha$ in the dynamics of the cost factor $\eta_i(t)$ is set to 0.05.

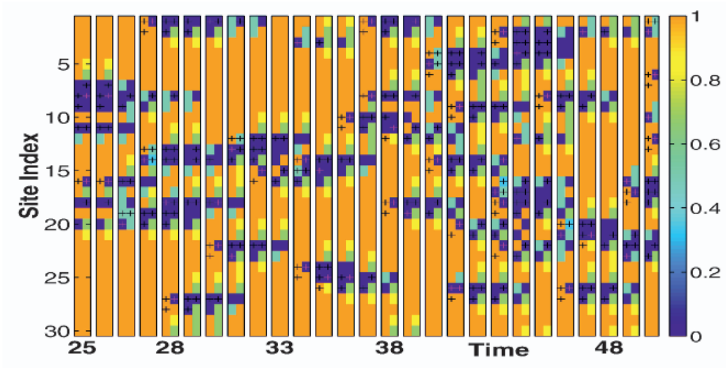


**Fig. 3.** Time evolution of output of the production network with degree $k = 2$. Zoom on a time interval. Each frame represents the production network at a given time step with index of the node on the $y$ axis and production layers on the $x$ axis. The layer of primary producers is now the left column of each frame, while the layer of retailers in the consumer market is the right column of each frame. Output is normalized in each frame by its maximum value in order to emphasize the relative spatial distribution and is represented by a color scale as specified by the color bar. Magenta crosses indicate production defaults occurring stochastically with probability $q$, while black crosses indicate bankruptcy (see text for more details)

In figures 3, 4, 5, the evolution of output over the production network is shown in an interval of 25 time steps. In order to follow the propagation of bankruptcies, we choose a represent different from the one used in fig. 1. Each frame represents the production network at a given time step with index of the node on the $y$ axis and production layers on the $x$ axis. The layer of primary producers is now the left column of each frame, while the layer of retailers in the consumer market is the right column of each frame. Output is normalized in each frame by its maximum value in order to emphasize the

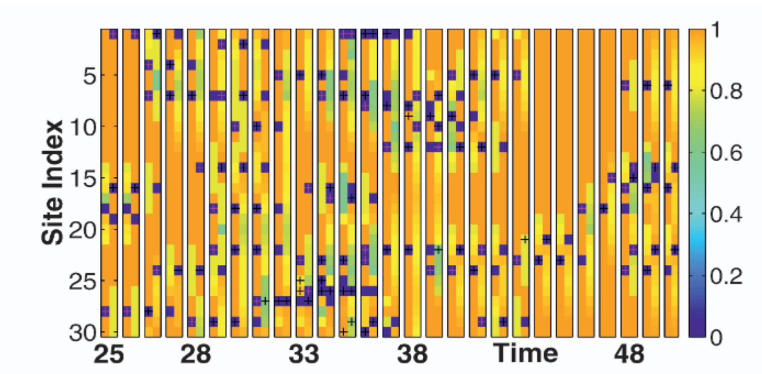


**Fig. 4.** Time evolution of output of the production network with degree $k = 5$. The figure is constructed in the same way as figure 3

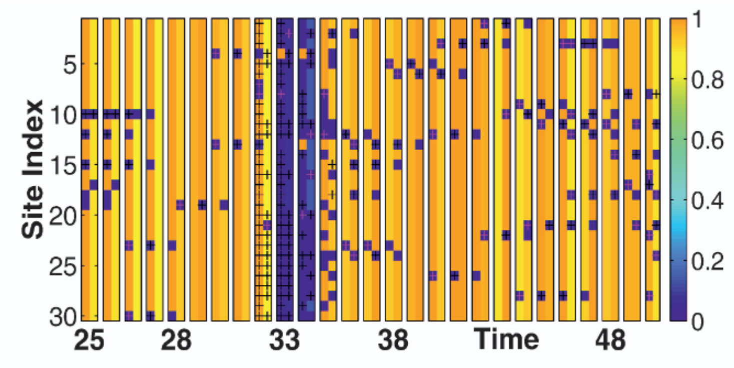


**Fig. 5.** Time evolution of output of the production network with degree $k = 5$. The figure is constructed in the same way as figure 3

relative spatial distribution and is represented by a color scale as specified by the color bar. Magenta crosses indicate production defaults occurring stochastically with probability $q$, while black crosses indicate bankruptcy. Following the position of the crosses from one frame to the next, it is possible to observe the propagation of bankruptcies over time. With the parameters chosen in this specific setting, production default in a firm also implies its bankruptcy, although it is not true in general.

With degree $k = 1$ (not shown), production is organized in chains, which are obviously very fragile to shocks, as the production default of a supplier/customer implies also the bankruptcy of its only customer/supplier. With the chosen values for cost, $c_s = 0.3$ and $c_r = 0.3$, and with degree $k = 2$, the default of a supplier is very likely to cause, the bankruptcy of its two customers, which in turn do not pay their suppliers, causing two additional bankruptcies. Overall, five firms go bankrupt in such an event, while with degree $k >= 3$ instead, the default of a supplier is very likely not to cause any bankruptcy. Simulations shown in figures 3, 4 confirm this estimate, although some deviations are possible, due the internal dynamics of the cost factor. With high degree ($k = 30$, 5) the feedback mechanism prevails on the risk diversification and larger avalanches occur (at time 32 in the figure).

The effect can be seen also in figures 6, 7, 8, where the evolution of output $Y_i(t)$ and cost factor $\eta_i(t)$ is shown over 100 time step. In order to emphasize the spatio-temporal patterns we use now another representation with respect to figures 3, 4, 5. The $x$ axis is time, while the $y$ axis represents the index of the nodes from 1 to N. In other words, positions from 1 to $n$ on $y$ (from to the top) represent the nodes of the first layer ($n = 30$ in this example), while positions from $n$ to $2n$ (from to the top) represent the nodes of the second layer. We chose $n$ and time interval relatively small to make the patterns visible. Output and cost factor are represented by a color scale as specified by the color bar. Magenta crosses indicate production defaults (occurring

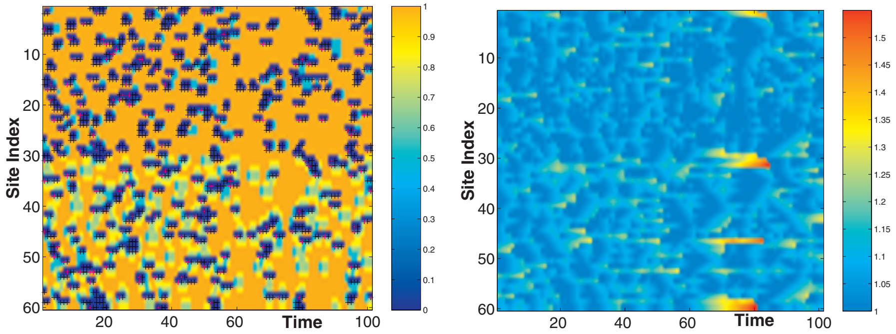


**Fig. 6.** Time evolution of output $Y_i(t)$ (**left**) and cost factor $\eta_i(t)$ (**right**) with degree $k = 2$. The $x$ axis is time, while the $y$ axis represents the index of the nodes from 1 to N. Output and cost factor are represented using a color scale specified by the color bar. Magenta crosses indicate production defaults, while black crosses indicate bankruptcy

stochastically with probability $q$), while black crosses indicate bankruptcy. Following the position of the crosses from one frame to the next, it is possible to observe the propagation of bankruptcies over time.

An important aspect of the phenomenon investigated here is that the instability induced at high $k$ is also visible at the aggregate level. In figure 9 the aggregate output of the network is shown for 200 time steps. Going from degree $k = 2$ (blue curve) to $k = 5$ (green curve) and to $k = 30$ (magenta curve), fluctuations first decrease and then increase.

A more detailed investigation of the trade off between individual risk diversification and systemic risk is out of the scope of this work and can be found in [16].

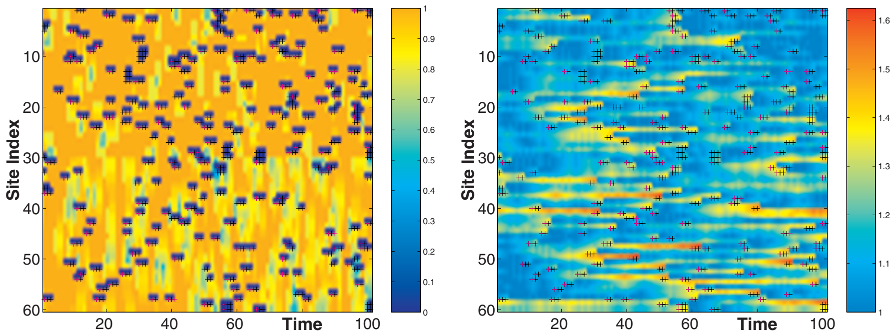


**Fig. 7.** Time evolution of output $Y_i(t)$ (**left**) and cost factor $\eta_i(t)$ (**right**) with degree $k = 5$. The figure is constructed in the same way as figure 6

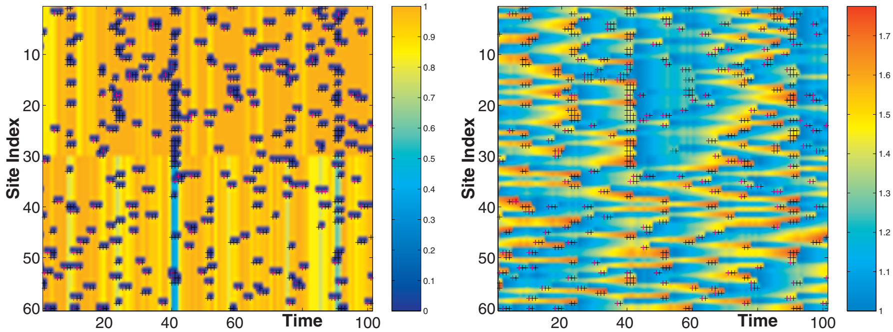


**Fig. 8.** Time evolution of output $Y_i(t)$ (**left**) and cost factor $\eta_i(t)$ (**right**) with degree $k = 30$. The figure is constructed in the same way as figure 6

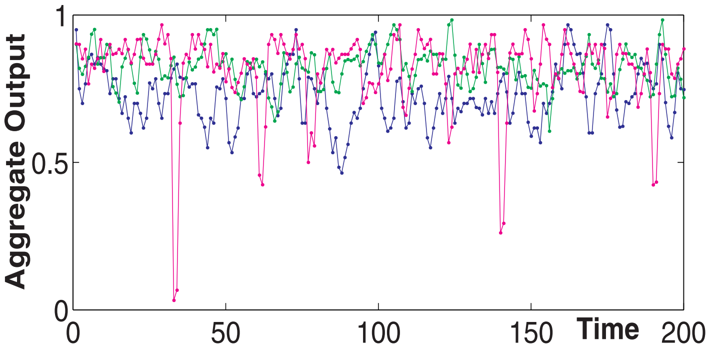


**Fig. 9.** Time evolution of aggregate output of the production network with different value of degree $k$: $k = 2$ (blue), $k = 5$ (green) and $k = 30$ (magenta)

## 3 Conclusion

In this chapter we have examined the impact of network density on the systemic risk in networks of credit relations. A node in a credit network may form several links in order to diversify its risk, but this may also induce a form of financial contagion.

Systemic risk raises today growing regulatory concern and policy makers would like to know how to ensure adequate fire walls in order to prevent the spill over of a crisis across institutions and firms. Avalanches of failures in networks have been studied extensively in the Complex System literature in the context of SOC and epidemic spreading, but, outside such two contexts, the investigation deserve more attention. In the economic literature, there is a growing body of work on systemic risk in credit networks, although there seems to be a dominant view on the positive role of the network density on the systemic risk.

Credit is extended by banks to firms (loans), by one bank to another (interbank credit) and by one firm to another (trade credit). In this chapter, we have focused on the last case and we have investigated a specific setting of the model introduced by [16]. In such a setting it is possible to isolate the impact of network density on the dynamics of avalanches.

Using a complex system approach, we have inferred some properties of the economic model based on a mean field approximation of another model, actually much simpler, that captures some essential dynamical features of the original one. We have shown how, under some conditions on the parameter chosen, a trade off emerges between individual risk diversification and systemic risk. This result is in line with recent finding in the economic literature, but it is in contrast with the dominant view. This works contributes to the debate on what are the appropriate regulations to ensure the robustness of credit networks.

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